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Spyridon Filippas

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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QUANTITATIVE UNIQUE CONTINUATION FOR WAVE OPERATORS WITH A JUMP DISCONTINUITY ACROSS AN INTERFACE AND APPLICATIONS TO APPROXIMATE CONTROL

by Spyridon Filippas

Abstract. — In this article we prove quantitative unique continuation results for wave operators of the form $\partial_t^2 - \operatorname{div}(c(x)\nabla\cdot)$, where the scalar coefficient c is discontinuous across an interface of codimension one in a bounded domain or on a compact Riemannian manifold. We do not make any assumptions regarding the geometry of the interface or the sign of the jumps of the coefficient c. The key ingredient is a local Carleman estimate for a wave operator with discontinuous coefficients. We then combine this estimate with the recent techniques of Laurent–Léautaud [21] to propagate local unique continuation estimates and obtain a global stability inequality. As a consequence, we deduce the cost of the approximate controllability for waves propagating in this geometry.

Résumé (*Prolongement unique quantitatif pour des équations d'onde avec des sauts à travers une interface et applications à la théorie du contrôle*). — Dans cet article nous démontrons des résultats de prolongement unique quantitatif pour des opérateurs d'onde de la forme $\partial_t^2 - \operatorname{div}(c(x)\nabla\cdot)$ où le coefficient scalaire c est discontinu à travers une interface de codimension un dans un domaine borné ou dans une variété riemannienne compacte. Nous ne faisons aucune hypothèse sur la géométrie de l'interface ou sur le signe du saut de c. L'ingrédient clef est une estimée de Carleman locale pour un opérateur d'onde ayant des coefficients discontinus. Nous combinons alors cette estimée avec les récentes techniques de Laurent-Léautaud [21] pour propager des estimées de prolongement unique locales et obtenir une estimée de stabilité globale. Comme conséquence, nous obtenons le coût de la contrôlabilité approchée pour les ondes se propageant dans cette géométrie.

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Spyridon Filippas, University of Helsinki, Yliopistonkatu 4, 00100 Helsinki, Finland •
E-mail : `spyridon.filippas@helsinki.fi`

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1. Introduction

For a wave operator P, the question of unique continuation consists in asking whether a partial observation of a wave on a small set $\omega \subset \Omega$ is sufficient to determine the *whole* wave. If this property holds, then the next natural question is whether we can *quantify* it. This is expressed via a stability estimate of the form

$$\|u\|_\Omega \lesssim \phi\left(\|u\|_\omega, \|Pu\|_\Omega, \|u\|_\Omega\right), \tag{1}$$

with ϕ satisfying

$$\phi(a,b,c) \overset{a,b\to 0}{\longrightarrow} 0, \quad \text{with } c \text{ bounded.}$$

Such estimates have numerous applications in control theory, spectral geometry, and inverse problems. Concerning the wave operator, a seminal unique continuation result was obtained by Robbiano in [36] and refined by Hörmander in [18]. The optimal version of this *qualitative* result was finally attained in the so-called Tataru, Hörmander, Robbiano–Zuily theorem [40, 41, 20, 38]. This theorem in fact deals with the more general case of operators with partially analytic coefficients and, in the particular case of a wave operator with coefficients independent of time, gives uniqueness across any noncharacteristic hypersurface. Recently, in [21], the authors proved a quantitative version of the latter theorem which, for the wave equation, is optimal with respect to the observation time and the stability estimate obtained. Note that a *qualitative* uniqueness result is equivalent to an approximate controllability result, and a *quantified* version of it gives an estimate of the control cost. The quantitative unique continuation result of [21] applies to (variants of) the operator $\partial_t^2 - \Delta_g$, where Δ_g is an elliptic operator with C^∞ coefficients. See also [5] for a related set of estimates concerning the wave operator.

However, in many contexts, waves propagate through singular media and therefore in the presence of nonsmooth coefficients, e.g., in the case of seismic waves [39] or acoustic waves [42, 1, 7] propagating through the Earth's crust. Models proposing to describe such phenomena use discontinuous metrics and, more precisely, metrics which are piecewise regular but present jumps along some hypersurfaces. See for instance the Mohorovičić discontinuity between the Earth's crust and the mantle. Another example arises in medical imaging. The human brain [33, 35] has two main components: white and gray matter. These have very different electrical conductivities, and models describing the situation are very similar to the preceding example.

The question of quantitative unique continuation across a jump discontinuity seems to be well understood in the elliptic/parabolic context. One of the first results (in the parabolic case) was [11], where the operator $\partial_t - \operatorname{div}(c\nabla\cdot)$ is studied with a monotonicity assumption imposed on the scalar coefficient $c = c(x)$: the observation should take place in the region where the coefficient c is smaller. In this article a global Carleman estimate was proved. Later, in the elliptic case in [28], a similar result was obtained but without any restriction on the sign of the jump of the coefficient. These techniques were extended to the parabolic context in [29]. The most recent (and general) result, to the best of our knowledge, was proved by Le Rousseau and Lerner [27], where the anisotropic case $(-\operatorname{div}(A(x)\nabla\cdot)$, with A a matrix jumping across an interface, is treated. Other interesting works in the elliptic context include [10, 12]. Therein, the problem of reducing the regularity of the leading coefficients away from the interface has been addressed.

The question of *exact* control for waves with jumps at an interface has already been addressed in the book [34]. A controllability result is proved for the operator $\partial_t^2-\operatorname{div}(c\nabla\cdot)$, with c a piecewise constant coefficient under a geometric assumption on the jump hypersurface and a sign condition on the jump. One of the first Carleman estimates was proved in the discontinuous setting in [4]. With the same assumption on the coefficient and assuming that the interface is convex, the authors prove linear quantitative stability estimates. Recently, in [3], quantitative results were also proved for interfaces that interpolate between star-shaped and convex. Other related works are [13] and [6].

However, to our knowledge, the question of stability estimates without any particular geometric assumption regarding the interface has not been studied yet. This is the main objective of this article.

1.1. Setting and statement of main results. — Let $(\mathcal{M}, g)$ be a smooth connected compact n-dimensional Riemannian manifold with or without boundary. We consider S an $(n-1)$-dimensional submanifold of $\mathcal{M}$ without boundary. We assume that $\mathcal{M}\backslash S = \Omega_- \cup \Omega_+$ with $\Omega_- \cap \Omega_+ = \emptyset$.

We consider a scalar coefficient $c(x) = \mathbb{1}_{\Omega_-} c_-(x) + \mathbb{1}_{\Omega_+} c_+(x)$ with $c_\pm \in C^\infty(\overline{\Omega}_\pm)$ satisfying $0 < c_{\min} < c(x) < c_{\max}$ uniformly on $\Omega_- \cup \Omega_+$ to ensure ellipticity. We shall work with the wave operator P defined as

$$P = \partial_t^2 - \operatorname{div}_g(c(x)\nabla_g), \quad \text{on } \mathbb{R}_t \times \Omega_- \cup \Omega_+. \tag{2}$$

We consider for $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ the following evolution problem:

$$\begin{cases} Pu = 0 & \text{in } (0,T) \times \Omega_- \cup \Omega_+ \\ u_{|S_-} = u_{|S_+} & \text{in } (0,T) \times S \\ (c\partial_\nu u)_{|S_-} = (c\partial_\nu u)_{|S_+} & \text{in } (0,T) \times S \\ u = 0 & \text{in } (0,T) \times \partial\mathcal{M} \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \mathcal{M}, \end{cases} \tag{3}$$

where we denote by ∂_ν a nonvanishing vector field defined in a neighborhood of S, normal to S (for the metric g), pointing into Ω_+, and normalized for g. We denote as well by $u_{|S_\pm}$ the traces of $u_{|\Omega_\pm}$ on the hypersurface S.

Notice that there are two extra equations in our system. These are some natural transmission conditions that we impose on the interface. These conditions imply that the underlying elliptic operator is self-adjoint on its domain and one can show using classical methods (for instance with the Hille–Yosida theorem) that the system (3) is well posed. For more details on this, we refer to Section 2.

Our first result provides a quantitative unique continuation result from an observation region ω for the discontinuous wave operator P.

In Section 1.3 we introduce $\mathcal{L}(\mathcal{M},\omega) = \sup_{x\in\mathcal{M}} \operatorname{dist}(x,\omega)$, the "largest distance" of the subset ω to a point of $\mathcal{M}$, where dist is a distance function adapted to $(\mathcal{M}, g, c)$.

Theorem 1.1. — *Consider $(\mathcal{M}, g), S, \Omega_\pm$ and P as defined in (2). Then for any nonempty open subset ω of $\mathcal{M}$ and any $T > 2\mathcal{L}(\mathcal{M},\omega)$, there exist C, κ, μ_0 such that for any $(u_0,u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and u solving (3) one has, for any $\mu \geq \mu_0$,*

$$\|(u_0,u_1)\|_{L^2\times H^{-1}} \leq Ce^{\kappa\mu} \|u\|_{L^2((0,T)\times\omega)} + \frac{C}{\mu} \|(u_0,u_1)\|_{H^1\times L^2} .$$

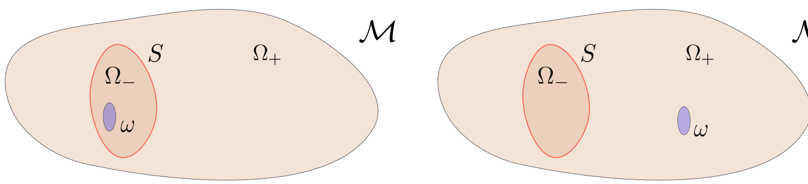
If, moreover, $\partial\mathcal{M} \neq \emptyset$ and Γ is a nonempty open subset of $\partial\mathcal{M}$, for any $T > 2\mathcal{L}(\mathcal{M},\Gamma)$, there exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0,u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and u solving (3), we have

$$\|(u_0,u_1)\|_{L^2\times H^{-1}} \leq Ce^{\kappa\mu} \|\partial_{\nu_\Gamma} u\|_{L^2((0,T)\times\Gamma)} + \frac{C}{\mu} \|(u_0,u_1)\|_{H^1\times L^2} .$$

Remark 1.2. — In fact, one can take $\mu > 0$ in the statement of the above theorem. However, we prefer to state it in this way in order to stress the fact that this estimate is interesting only when μ is large.

With the above, one can recover the following qualitative result: "If we do not see anything from ω during a time T strictly larger than $2\mathcal{L}(\mathcal{M},\omega)$, then there is no wave at all." Indeed, if $\|u\|_{L^2((0,T)\times\omega)} = 0$, then letting $\mu \to +\infty$ in the above inequality implies that $(u_0,u_1) = 0$.

An important aspect of this theorem is that there is no assumption regarding the sign of the jump of the coefficient c and, consequently, the observation region ω can be chosen indifferently on Ω_- or Ω_+. Let us explain why this is quite surprising. Suppose, to fix ideas, that $c_- < c_+$ are two constants. We can then interpret $\sqrt{c_-}$ and $\sqrt{c_+}$ as the speed of propagation of a wave traveling through two isotropic media, Ω_- and Ω_+, with different refractive indices, n_- and n_+, respectively (recall that $n_\pm = 1/\sqrt{c_\pm}$). Imagine that a wave starts traveling from a region that is inside Ω_-. One has $\frac{\sqrt{c_-}}{\sqrt{c_+}} = \frac{n_+}{n_-}$ and therefore the assumption $c_- < c_+$ translates into $n_- > n_+$. Then, Snell–Descartes law states that when a wave travels from a medium with a higher refractive index to one with a lower refractive index, there is a critical angle from which there is *total internal reflection*, that is, no refraction at all. At the level of geometric optics, that is to say in the high-frequency regime, such a wave stays trapped inside Ω_-. Therefore one expects that, at least at high frequency, no information propagates from Ω_- to Ω_+, following the laws of geometric optics. Our result (see Theorem 1.3) states that the intensity of waves in Ω_+ is at least exponentially small in terms of the typical frequency Λ of the wave.



(A) The observation takes place inside Ω_-.

(B) The observation takes place inside Ω_+. If $c_- < c_+$, then a part of the wave may be trapped inside Ω_-. Nevertheless, the quantitative unique continuation and its consequences still hold.

We can reformulate Theorem 1.1 in a manner closer to quantitative estimates such as (1). Indeed, optimizing the inequalities of Theorem 1.1 with respect to μ yields the following result (which we state only for the interior observation case):

Theorem 1.3. — *Under the assumptions of Theorem 1.1 there exists $C > 0$ such that for all $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ with $(u_0, u_1) \neq (0, 0)$ one has*

$$\begin{aligned} \|(u_0,u_1)\|_{H^1\times L^2} &\le Ce^{C\Lambda}\, \|u\|_{L^2((0,T)\times\omega)}, \\ \|(u_0,u_1)\|_{L^2\times H^{-1}} &\le C\frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\log\left(1+\frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|u\|_{L^2((0,T)\times\omega)}}\right)}, \end{aligned} \tag{4}$$

where $\Lambda = \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|(u_0,u_1)\|_{L^2\times H^{-1}}}$.

Note that Λ represents the typical frequency of the initial data. Theorem 1.3 is a direct consequence of Theorem 1.1 and Lemma A.3 in [21]. Notice that the function

$$x \mapsto \frac{1}{\log(1+1/x)},$$

appearing on the right hand side of (4) has been tacitly extended by continuity by 0 when $x = 0$.