SINGULARITIES IN MECHANICS: FORMATION, PROPAGATION AND MICROSCOPIC DESCRIPTION

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***Abstract.* —** This edition of "Panoramas & Synthèses" follows the thematic trimester "Singularities in Mechanics" that was organized at the Institut Henri Poincaré during winter 2008 by Jens Eggers, Christophe Josserand and Laure Saint-Raymond.

The central questions – which are discussed in a transverse way – are the formation, the propagation and the microscopic description of singularities. This volume gathers different articles showing the variety of mathematical approaches and physical problems, ranging from surface singularities in fluid mechanics to self-similar solutions of the Non Linear Schrödinger equation. In fluid mechanics, the wave breaking dynamics is investigated (Yves Pomeau and Martine Le Berre), the evolution of vortex filaments (Valeria Banica and Luis Vega) as well as the formation and cusps in surface flows (Jens Eggers and Marco Fontelos) while the mathematical grounds of such singularities are described for capillary flows (Antoine Mellet) and two dimensional surface flows (Claude Bardos and David Lannes).

***Résumé* (Singularités en Mécanique : formation, propagation et description microscopique)**

Cette édition de « Panoramas et Synthèses » fait suite au trimestre thématique « Singularités en Mécanique » organisé à l'Institut Henri Poincaré pendant l'hiver 2008 par Jens Eggers, Christophe Josserand et Laure Saint-Raymond. Les questions centrales – qui y sont abordées de façon transverse – sont la formation, la propagation et la description microscopique des singularités. Les différents articles réunis dans cette revue illustrent la variété des méthodes mathématiques utilisées et des problèmes physiques concernés, des singularités d'interfaces en mécanique des fluides aux solutions auto-similaires de l'équation de Schrödinger nonlinéaire.

Les problèmes abordés concernent le déferlement des vagues (Yves Pomeau et Martine Leberre), l'évolution des filaments de vortex (Valeria Banica et Luis Vega), la formation de cusps en mécanique des fluides (Jens Eggers et Marco Fontelos), les fondements mathématiques de ces singularités étant abordés dans le cas de la capillarité (Antoine Mellet) et des instabilités des interfaces (Claude Bardos et David Lannes).

CONTENTS

FOREWORD

The contributions gathered in this volume correspond to courses and series of lectures that have been delivered at the Institut Henri Poincaré in 2008, in the occasion of the interdisciplinary program "Singularities in mechanics : formation, propagation, microscopic description". By "**singularities**" we mean all phenomena involving strong focusing or rapid oscillations, leading to non-smooth behavior in some continuum description.

The study of such behaviors is of crucial importance from an experimental, as well as from a theoretical and numerical point of view. The issue is to understand whether a **continuum description**, which is valid only away from any singular set, is still able to deliver a coherent description of the emergence of a singularity, and perhaps even of the dynamics after, in the presence of a singularity.

The study of singularities is well established in solid mechanics (cracks), in both compressible and incompressible fluid dynamics (shocks, vortices), and is central for problems involving fronts and interfaces (drops and bubbles, surface instabilities). The precise description of a singularity will certainly be different in different systems, but a number of **qualitative properties** (such as the type of self-similar behavior), as well as methods of mathematical analysis (such as the definition of functional spaces which define which singularities are admissible) should be applicable in a much more general fashion.

In order to **identify the different issues** we will deal with in the sequel, we start by considering a very **simple example** of equation for which explicit computations allow to understand the structure and the dynamics of singular solutions. This model has no direct physical application, but it is a prototype of monodimensional hyperbolic systems of conservation laws which arise for instance in fluid mechanics or in elasticity.

Then, general considerations on the characterization of singularities will be discussed in order to introduce the different contributions of this volume.

1. Singular solutions of the Hopf equation

The Hopf equation is a scalar, one-dimensional equation, meaning that the unknown u is real and depends on time $t \in \mathbf{R}^+$ and on one space variable $x \in \mathbf{R}$.

It states

$$\partial_t u + \partial_x \Big(\frac{1}{2} u^2 \Big) = 0 \tag{1.1}$$

or equivalently for classical solutions

$$\partial_t u + u \, \partial_x u = 0. \tag{1.2}$$

This last equation is also called the inviscid Burgers equation and can be deduced in many physical systems where transport is crucial, such as film dynamics [**12**], shock waves, front dynamics [**4, 10**] or traffic jam modeling for instance (see also [**2, 9, 11**] for general reviews). Using this last form, one can compute explicitly the solution in terms of the initial data

$$u_{|t=0} = u_0,$$

at least for small times. For general transport equations, one indeed has the so-called method of characteristics which is described in the next paragraph.

1.1. The method of characteristics. — *The solutions to the transport equation*

$$\partial_t v + a(t,x) \, \partial_x v = 0, \quad v_{|t=0} = v_0 \tag{1.3}$$

can be written simply in terms of the solutions to the ordinary differential equation

$$\frac{dX}{dt} = a(t,X), \quad X(0,x_0) = x_0. \tag{1.4}$$

We indeed have

$$v(t, X(t,x_0)) = v_0(x_0).$$

If $X_t : x \mapsto X(t,x)$ is a bijection, then

$$v(t,x) = v_0(X_t^{-1}(x)).$$

In the case of a constant convection field a, the motion is uniform (see Fig. 1)

$$v(t,x) = v_0(x - at).$$

Under suitable regularity assumptions on a, the Cauchy-Lipschitz theorem ensures that the trajectories of (1.4) are locally well-defined and unique, so that X_t is invertible (see Fig. 2).

In the case of the Hopf equation, the velocity field u is transported by itself. We thus have

$$\frac{dX}{dt} = u(t,X), \quad X(0,x_0) = x_0,$$

$$u(t, X(t,x)) = u_0(x).$$

as long as X is a diffeomorphism, *i.e.*, a smooth change of variables. Note that the fact that X is a diffeomorphism is related to the regularity of u (by the Cauchy-Lipschitz theorem).

Differentiating the equation of characteristics with respect to x, we get

$$\frac{d}{dt}\frac{dX}{dx} = \frac{d}{dx}(u(t,X)) = u_0'(x).$$

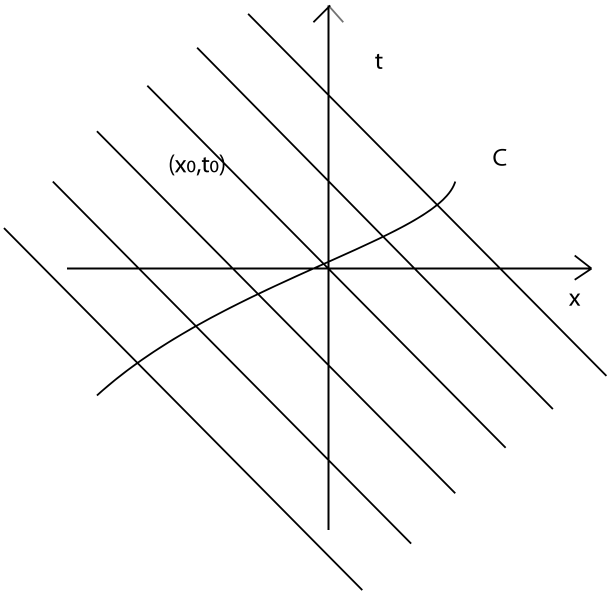

FIGURE 1. Characteristics are straight lines

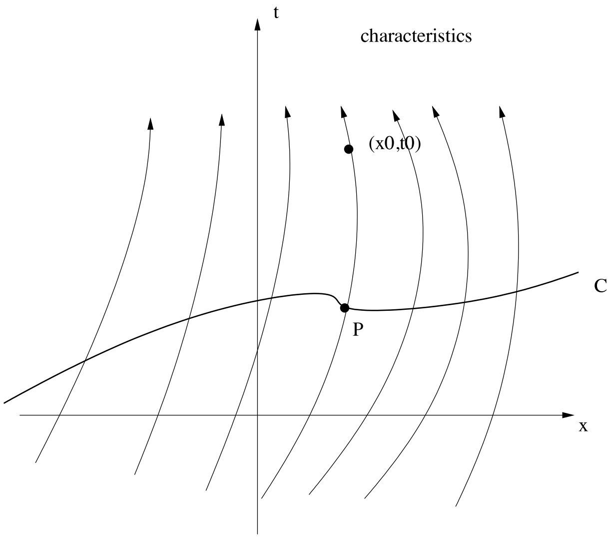

FIGURE 2. Characteristics give a mapping from **R** to **R**

Integrating with respect to t, we obtain

$$\frac{dX}{dx} = 1 + u_0'(x)t.$$

At time

$$t_0 = \frac{1}{\sup_{x\in\mathbf{R}}(-u_0'(x))_+}$$

the function $\frac{dX}{dx}$ cancels at least in one point, so that X is no more a diffeomorphism.

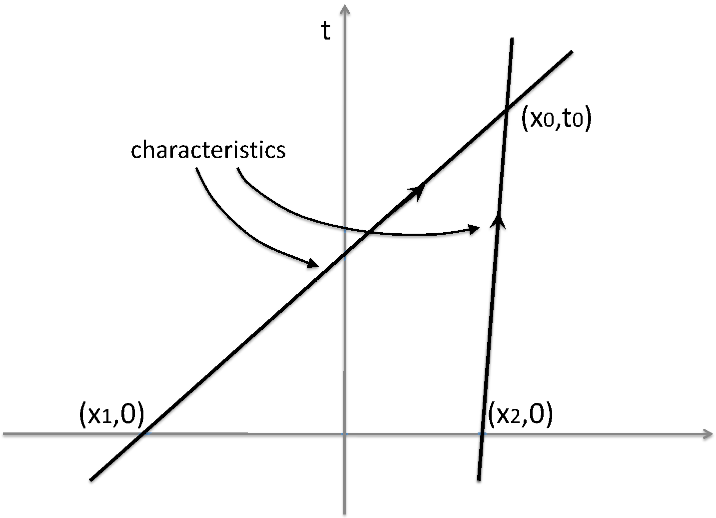

FIGURE 3. Crossing of characteristics

A singularity appears at time t_0. Indeed, at point (t_0, x_0) the solution u becomes multivalued and thus $u(t_0, x_0)$ is not defined. Furthermore, there is a jump discontinuity

$$\lim_{x\to x_0^-} u(t_0, x) \neq \lim_{x\to x_0^+} u(t_0, x).$$

In other words, t_0 is the time of breakdown, corresponding to the formation of a singularity.

It is then natural to ask whether the solutions can be defined in some weaker sense for later times.

1.2. Weak solutions, entropy and uniqueness. — *A solution in the sense of distributions* of the Hopf equation is any function $u \in L^\infty(\mathbf{R}^+ \times \mathbf{R})$ (defined and bounded almost everywhere) such that for all $\phi \in C_c^\infty(\mathbf{R}^+ \times \mathbf{R})$

$$\iint \Big(u\,\partial_t\phi + \frac{1}{2}u^2\,\partial_x\phi\Big)\,dx\,dt = \int u_0\phi_{|t=0}\,dx.$$

Remarks

(i) *Note that, because u is a function (defined and bounded almost everywhere), u^2 is defined : it is a function defined and bounded almost everywhere.*

(ii) *Note also that, in distributional sense, both formulations (1.1) and (1.2) of the Hopf equation are not equivalent : we will use the conservative form (1.1).*

With that notion of solution, we have no more uniqueness! Starting for instance from the Heaviside function $u_0 = H$, we can check that both functions u_1 and u_2 defined respectively by

$$u_1(t,x) = H\Big(x - \frac{1}{2}t\Big),$$

$$u_2(t,x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x}{t} \text{ if } 0 < x < t \\ 1 \text{ if } x > t \end{cases}$$

and represented on Fig. 4, are solutions in the sense of distributions of the Hopf equation, *i.e.*,

$$\iint \Big(u\,\partial_t\phi + \frac{1}{2}u^2\,\partial_x\phi\Big)\,dx\,dt = \int u_0\phi_{|t=0}\,dx.$$

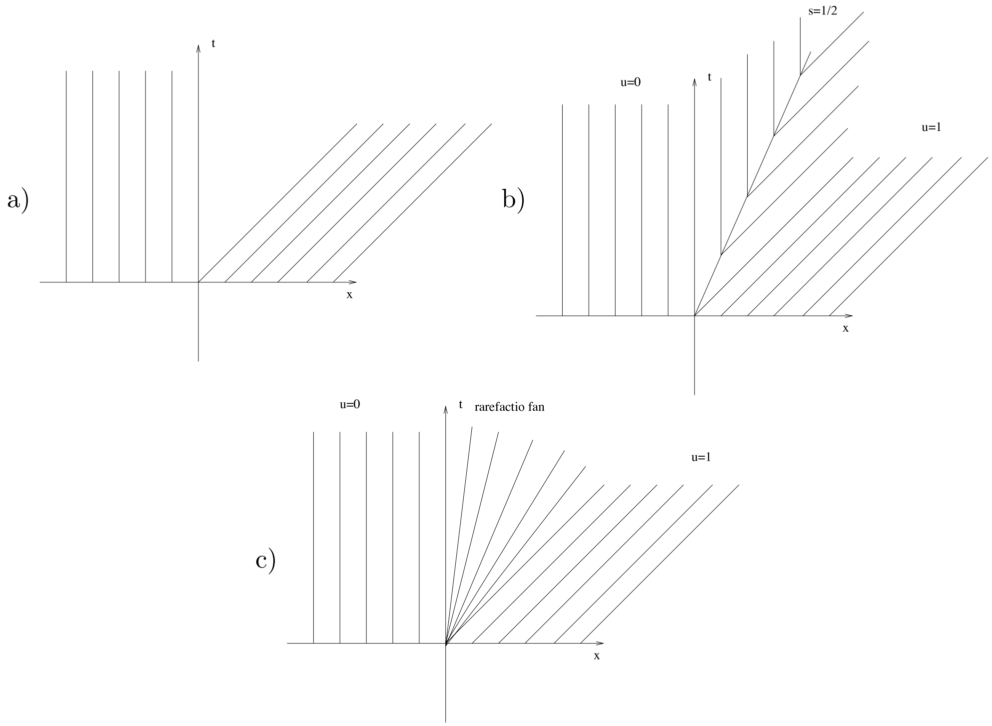

FIGURE 4. Non uniqueness of distributional solutions: a) characteristics associated to H b) solution u_1 c) solution u_2

Physically relevant solutions are selected by imposing further conditions, which should especially guarantee uniqueness.

Consider the characteristics of our problem, and consider a shock, *i.e.*, a discontinuity propagating at speed s that satisfies the jump condition

$$-s[u]_-^+ + \frac{1}{2}[u^2]_-^+ = 0.$$

Through every point of the shock, one can draw two characteristics, one of each side of the shock. Either both of them can be traced back to the initial line (see Fig. 5a), or both can be traced upwards to the future (see Fig. 5b). A shock is said to satisfy the entropy condition in the first case. Shocks of the second kind, called rarefaction shocks, are not admissible since they are not determined by the initial data (causality principle) [**1**].

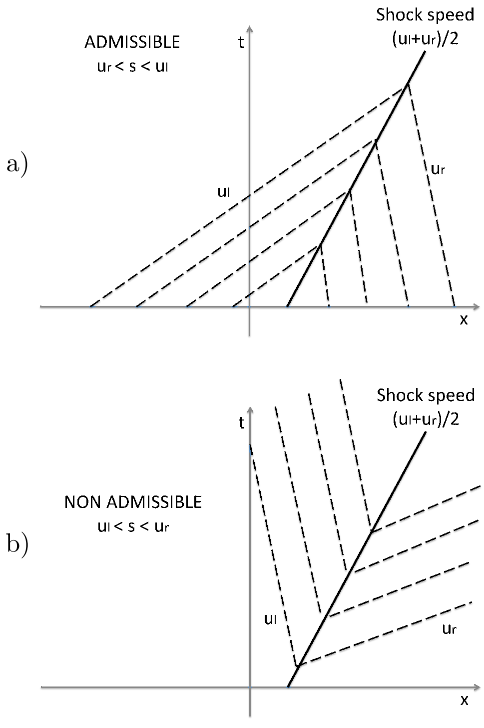

FIGURE 5. Entropy condition

One can prove (using the weak formulation of the equation and integrations by parts outside from shock lines) that the energy $\int u^2(t,x)\,dx$ cannot increase for a weak solution whose shocks satisfy the entropy condition. Admissible solutions actually satisfy

$$\partial_t u^2 + \partial_x \Big(\frac{2}{3} u^3\Big) \leq 0$$

which expresses the second principle of thermodynamics, namely the fact that the entropy increases. Both criteria are actually equivalent.

With this additional constraint, we will retrieve the uniqueness of the solutions. In other words, the system

$$\left.\begin{aligned} \partial_t u + \frac{1}{2}\partial_x u^2 = 0 \\ \partial_t u^2 + \frac{2}{3}\partial_x u^3 \leq 0 \end{aligned}\right\} \quad \text{in the sense of distributions} \tag{1.5}$$

captures the dynamics beyond the apparition of the singularities.

Remark. — *We will see in the next paragraph that the entropy condition is inherited from the microscopic structure of shocks, starting from a suitable kinetic model and considering its fast relaxation limit as the mean free-path tends to 0.*

1.3. A microscopic model. — We introduce the microscopic distribution of particles $f \equiv f(t,x,v)$ which at time t are located at x and have velocity v [**6, 5**]. The kinetic equation we will consider states

$$\partial_t f + v\partial_x f = \frac{1}{\epsilon}(\mathbf{1}_{[0,u]} - f) \text{ with } u(t,x) = \int f(t,x,v)\, dv.$$

If $u < 0$ we abuse the notation $[0,u]$ to denote $[u,0]$.

That equation expresses some balance between the free transport (left-hand side) and some collision process (right-hand side). Note that we consider here only the global effect of collisions leading to some relaxation towards local thermodynamic equilibrium, which is similar to the BGK model of the Boltzmann equation for perfect gases.

We therefore expect that

– in the fast relaxation limit $\epsilon \to 0$ local thermodynamic equilibrium should be reached almost everywhere $f(t,x,v) = \mathbf{1}_{[0,u(t,x)]}(v)$ with u solution to the Hopf equation;
– in the vicinity of macroscopic discontinuities the effect of the transport is non negligible : the microscopic density should remain smooth with a spatial derivative of order $O(\epsilon^{-1/2})$.

An explicit integral representation of the solution is obtained using Duhamel's formula.

Separating between the linear part and the nonlinear kernel, the kinetic equation may be rewritten

$$\partial_t f + v\, \partial_x f + \frac{1}{\epsilon} f = \frac{1}{\epsilon}\mathbf{1}_{[0,u]}.$$

The friction term leads to some exponential decay. Remarking that the characteristics of the free transport are given by $X(t,x,v) = x+tv$ and $V(t,x,v) = v$, and integrating with respect to time, we get

$$f(t, x+vt, v)e^{\frac{t}{\epsilon}} = f_0(x,v) + \int_0^t \left(\frac{1}{\epsilon}\mathbf{1}_{[0,u(t,x+vs)]}e^{\frac{s}{\epsilon}}\right) ds$$

or equivalently

$$f(t,x,v) = f_0(x-vt, v)e^{-\frac{t}{\epsilon}} + \int_0^t \left(\frac{1}{\epsilon}\mathbf{1}_{[0,u(t,x-v(t-s))]}e^{-\frac{(t-s)}{\epsilon}}\right) ds.$$

For fixed ϵ, the strategy to solve the kinetic equation is therefore

- to define a mapping T which at any microscopic distribution g associates the microscopic distribution $f = T(g)$ defined by the previous formula with macroscopic field $u = \int g\, dv$;
- to prove that this mapping T is a contraction in some suitable norm, namely the integral norm $L^\infty_t(L^1_{x,v})$;
- to conclude by Picard's theorem that there exists a unique fixed point for this mapping T, *i.e.*, a unique strong solution to the kinetic equation.

Note that, if we further assume that $f_0 \in L^\infty(\mathbf{R}_x \times \mathbf{R}_v)$ (defined and bounded almost everywhere), the integral representation combined with the trivial bound $\mathbf{1}_{[0,u]} \leq 1$ gives the maximum principle

$$\|f(t)\|_\infty \leq e^{-\frac{t}{\epsilon}}\|f_0\|_\infty + (1 - e^{-\frac{t}{\epsilon}}).$$

Some important features of the kinetic model can be deduced from the integral representation

Monotonicity. — Let $f_0, \tilde{f}_0$ be two integrable functions such that $f_0 \leq \tilde{f}_0$ almost everywhere. Denote by f and $\tilde{f}$ the solutions of the kinetic equation with respective initial data f_0 and $\tilde{f}_0$. Then $f \leq \tilde{f}$.

Finite speed of propagation. — Assume that $f_0 \in L^1 \cap L^\infty(\mathbf{R}_x \times \mathbf{R}_v)$ and that it is compactly supported, *i.e.*, $f_0(x,v) \equiv 0$ whenever $|x|^2 + |v|^2 \geq R^2$. Then $f(t)$ is compactly supported in v, and thus compactly supported in x.

Kinetic entropy functionals. — For any $h : (z,v) \mapsto |z - \mathbf{1}_{[0,K]}|$, one has

$$\partial_t \int h(f,v)\, dv + \partial_x \int h(f,v) v\, dv \leq 0$$

which is similar to Boltzmann's H theorem for the Boltzmann equation for perfect gases. It expresses some irreversibility of the dynamics, and is thus related to the second principle of thermodynamics.

The fast relaxation limit of this kinetic equation is actually described by weak solutions to the Hopf equation, supplemented with the entropy condition (1.5).