

## ABOUT THE BEHAVIOR OF REGULAR NAVIER-STOKES SOLUTIONS NEAR THE BLOW UP

BY EUGÉNIE POULON

---

ABSTRACT. — In this paper, we present some results about blow up of regular solutions to the homogeneous incompressible Navier-Stokes system, in the case of data in the Sobolev space  $\dot{H}^s(\mathbb{R}^3)$ , where  $\frac{1}{2} < s < \frac{3}{2}$ . Firstly, we will introduce the notion of minimal blow up Navier-Stokes solutions and show that the set of such solutions is not only nonempty but also compact in a certain sense. Secondly, we will state an uniform blow up rate for minimal Navier-Stokes solutions. The key tool is profile theory as established by P. Gérard [11].

### 1. Introduction

We consider the Navier-Stokes system for incompressible fluids evolving in the whole space  $\mathbb{R}^3$ . Denoting by  $u$  the velocity, a vector field in  $\mathbb{R}^3$ , by  $p$  in  $\mathbb{R}$  the pressure function, the Cauchy problem for the homogeneous incompressible Navier-Stokes system is given by

$$(1) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

---

*Texte reçu le 3 décembre 2014, modifié le 10 mars 2016, accepté le 10 mars 2016.*

EUGÉNIE POULON, Laboratoire Jacques-Louis Lions - UMR 7598, Université Pierre et Marie Curie, Boîte courrier 187, 4 place Jussieu, 75252 Paris Cedex 05, France •  
*E-mail* : [poulon@ann.jussieu.fr](mailto:poulon@ann.jussieu.fr)

Key words and phrases. — Navier-Stokes equations, blow up, profile decomposition.

Throughout this paper, we will adopt the useful notation  $NS(u_0)$  to mean the maximal solution of the Navier-Stokes system, associated with the initial data  $u_0$ .

DEFINITION 1.1. — Let  $s$  in  $\mathbb{R}$ . The homogeneous Sobolev space  $\dot{H}^s(\mathbb{R}^3)$  is the space of tempered distributions  $u$  over  $\mathbb{R}^3$ , the Fourier transform of which belongs to  $L^1_{loc}(\mathbb{R}^3)$  and satisfies

$$\|u\|_{\dot{H}^s} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

It is known that  $\dot{H}^s(\mathbb{R}^3)$  is an Hilbert space if and only if  $s < \frac{3}{2}$ . We will denote by  $(\cdot|\cdot)_{\dot{H}^s(\mathbb{R}^3)}$ , the scalar product in  $\dot{H}^s(\mathbb{R}^3)$ . From now on, for the sake of simplicity, it will be an implicit understanding that all computations will be done in the whole space  $\mathbb{R}^3$ .

Before stating the results we prove in this paper, we recall two fundamental properties of the incompressible Navier-Stokes system. The first one is the conservation of the  $L^2$  energy. Formally, let us take the  $L^2$  scalar product with the velocity  $u$  in the equation. We get

$$(2) \quad \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = - \int_{\mathbb{R}^3} (u \cdot \nabla u(t)|u(t))_{L^2} - \int_{\mathbb{R}^3} (\nabla p(t)|u(u))_{L^2}.$$

Thanks to the divergence free condition, obvious integration by parts implies that, for any vector field  $a$

$$(3) \quad (u \cdot \nabla a|a)_{L^2} = 0 = (\nabla p|a)_{L^2}.$$

This gives

$$(4) \quad \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0.$$

The second property of the system is the scaling invariance. Let us define the operator:

$$(5) \quad \forall \alpha \in \mathbb{R}^+, \forall \lambda \in \mathbb{R}_*^+, \forall x_0 \in \mathbb{R}^3, \quad \Lambda_{\lambda, x_0}^\alpha u(t, x) \stackrel{\text{def}}{=} \frac{1}{\lambda^\alpha} u\left(\frac{t}{\lambda^2}, \frac{x - x_0}{\lambda}\right).$$

If  $\alpha = 1$ , we note  $\Lambda_{\lambda, x_0}^1 = \Lambda_{\lambda, x_0}$ .

It is easy to see that if  $u$  is a smooth solution of Navier-Stokes system on  $[0, T] \times \mathbb{R}^3$  with pressure  $p$  associated with the initial data  $u_0$ , then, for any positive  $\lambda$ , the vector field and the pressure

$$u_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda, x_0} u \text{ and } p_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda, x_0}^2 p$$

is a solution of Navier-Stokes system on the interval  $[0, \lambda^2 T] \times \mathbb{R}^3$ , associated with the initial data

$$u_{0, \lambda} = \Lambda_{\lambda, x_0} u_0.$$

This leads to the definition of scaling invariant space, which is a key notion to investigate local and global well-posedness issues for Navier-Stokes system.

DEFINITION 1.2. — A Banach space  $X$  is said to be scaling invariant, if its norm is invariant under the scaling transformation defined by  $u \mapsto u_\lambda$

$$\|u_\lambda\|_X = \|u\|_X$$

The first main result on incompressible Navier-Stokes system is due to J. Leray, who proved in [19] in 1934 that given an initial data in the energy space  $L^2$ , the associated NS-solutions, called weak solutions, exist globally in time. The key ingredient of the proof is the  $L^2$ -energy conservation (4). Moreover, such solutions are unique in 2-D; but the uniqueness in 3-D is still an open problem. One way to adress this question of unique solvability in 3-D is to demand smoother initial data. In this case, we definitely get a unique solution, but the other side of coin is that the problem is only locally well-posed (and becomes globally well-posed under a scaling invariant smallness assumption on the initial data). J. Leray stated such a theorem of existence of solutions, which he called semi-regular solutions.

THEOREM 1.1. — *Let an initial data  $u_0$  be a divergence free vector field in  $L^2$  such that  $\nabla u_0$  belongs to  $L^2$ . Then, there exists a positive time  $T$ , and a unique solution  $NS(u_0)$  in  $C^0([0, T], \dot{H}^1) \cap L^2([0, T], \dot{H}^2)$ .*

*Moreover, a constant  $c_1$  exists such that if  $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \leq c_1$ , then  $T$  can be chosen equal to  $\infty$ .*

The reader will have noticed that the quantity  $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$  is scaling invariant under the operator  $\Lambda_{\lambda, x_0}$ . Actually, that is the starting point of many frameworks concerning the global existence in time of solutions under a scaling invariant smallness assumption on the data. The celebrated first one was introduced in 1964, by H. Fujita and T. Kato. These authors stated a similar result as J. Leray, but they demanded less regularity on the data. Indeed, they proved that for any initial data in  $\dot{H}^{\frac{1}{2}}$ , there exists a positive time  $T$  and there exists a unique solution  $NS(u_0)$  belonging to  $C^0([0, T], \dot{H}^{\frac{1}{2}}) \cap L^2([0, T], \dot{H}^{\frac{3}{2}})$ . Moreover, if  $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$  is small enough, then the solution is global in time. This theorem can be proved by a fixed-point argument and the key ingredient of the proof is that the Sobolev space  $\dot{H}^{\frac{1}{2}}$  is invariant under the operator  $\Lambda_{\lambda, x_0}$ . In other words, the Sobolev space  $\dot{H}^{\frac{1}{2}}$  has exactly the same scaling as Navier-Stokes equation. We refer the reader to [1], [7] or [18] for more details of the proof. But in this paper, we work with initial data belonging to homogeneous Sobolev spaces,  $\dot{H}^s$  with  $\frac{1}{2} < s < \frac{3}{2}$ , which means that we are above the natural scaling of the equation. The first thing to do is to provide an existence theorem of Navier-Stokes solutions with data in such Sobolev spaces  $\dot{H}^s$ . The Cauchy problem is known to be locally well-posed; it can be proved by a fixed-point

procedure in an adequate function space (we refer the reader to the book [18], from page 146 to 148, of P-G. Lemarié-Rieusset).

We shall constantly be using the following simplified notations:

$$L_T^\infty(\dot{H}^s) \stackrel{\text{def}}{=} L^\infty([0, T], \dot{H}^s) \text{ and } L_T^2(\dot{H}^{s+1}) \stackrel{\text{def}}{=} L^2([0, T], \dot{H}^{s+1}).$$

Let us define the relevant function space we shall be working with in the sequel:

$$X_T^s \stackrel{\text{def}}{=} L_T^\infty(\dot{H}^s) \cap L_T^2(\dot{H}^{s+1}), \text{ equipped with } \|u\|_{X_T^s}^2 \stackrel{\text{def}}{=} \|u\|_{L_T^\infty(\dot{H}^s)}^2 + \|u\|_{L_T^2(\dot{H}^{s+1})}^2.$$

**THEOREM 1.2.** — *Let  $u_0$  be in  $\dot{H}^s$ , with  $\frac{1}{2} < s < \frac{3}{2}$ . Then there exists a time  $T$  and there exists a unique solution  $NS(u_0)$  such that  $NS(u_0)$  belongs to  $L_T^\infty(\dot{H}^s) \cap L_T^2(\dot{H}^{s+1})$ .*

*Moreover, let  $T_*(u_0)$  be the maximal time of existence of such a solution. Then, there exists a positive constant  $c$  such that*

$$(6) \quad T_*(u_0) \|u_0\|_{\dot{H}^s}^{\sigma_s} \geq c, \text{ with } \sigma_s \stackrel{\text{def}}{=} \frac{1}{\frac{1}{2}(s - \frac{1}{2})}.$$

**REMARK 1.1.** — As a by-product of the proof of Picard’s Theorem, we get actually for free the following property: if the initial data is small enough (in the sense of there exists a positive constant  $c_0$ , such that  $T \|u_0\|_{\dot{H}^s}^{\sigma_s} \leq c_0$ ), then a unique Navier-Stokes solution associated with it exists (locally in time, until the blow up time given by the relation (6)) and satisfies the following linear control

$$(7) \quad \forall 0 \leq T \leq \frac{c_0}{\|u_0\|_{\dot{H}^s}^{\sigma_s}}, \|NS(u_0)(t, \cdot)\|_{X_T^s} \leq 2 \|u_0\|_{\dot{H}^s}.$$

Formula (6) invites us to consider the lower boundary, denoted by  $A_s^{\sigma_s}$ , of the lifespan of such a solution

$$A_s^{\sigma_s} \stackrel{\text{def}}{=} \inf \left\{ T_*(u_0) \|u_0\|_{\dot{H}^s}^{\sigma_s} \mid u_0 \in \dot{H}^s ; T_*(u_0) < \infty \right\}.$$

Obviously,  $A_s^{\sigma_s}$  exists and is a positive real number and we always have the formula

$$(8) \quad T_*(u_0) \|u_0\|_{\dot{H}^s}^{\sigma_s} \geq A_s^{\sigma_s}.$$

Throughout this paper, we make the assumption of blow up, which is still an open problem. More precisely, we claim the following hypothesis.

*Hypothesis  $\mathcal{H}$ :* We consider  $s$  in  $]\frac{1}{2}, \frac{3}{2}[$ , such that a divergence-free vector field  $u_0$  exists in  $\dot{H}^s$  with a finite the lifespan  $T_*(u_0)$ .

**DEFINITION 1.3 (Minimal blow up solution).** — We say that  $u = NS(u_0)$  is a minimal blow up solution if  $u_0$  satisfies

$$T_*(u_0) \|u_0\|_{\dot{H}^s}^{\sigma_s} = A_s^{\sigma_s}.$$

In other terms  $u = NS(u_0)$  is a minimal blow up solution if and only if  $A_s^{\sigma_s}$  is reached.

*Question:* Under Hypothesis  $\mathcal{H}$ , do some minimal blow up solutions exist?

We will prove a stronger result: the set of initial data generating minimal blow up solutions with blow up time  $T_*$ , denoted by  $\mathcal{M}_s(T_*)$ , is not only a nonempty subset of  $\dot{H}^s$  (which, in particular, gives the positive answer to the question) but also compact in a sense which is given in Theorem 1.3.

**THEOREM 1.3.** — *Assuming hypothesis  $\mathcal{H}$ , for any finite time  $T_*$ , the set  $\mathcal{M}_s(T_*)$  is non empty and compact, up to translations. This means that for any sequence  $(u_{0,n})_{n \in \mathbb{N}}$  of points in the set  $\mathcal{M}_s(T_*)$ , a sequence  $(x_n)_{n \in \mathbb{N}}$  of points of  $(\mathbb{R}^3)^\mathbb{N}$  and a function  $V$  in  $\mathcal{M}_s(T_*)$  exist such that, up to an extraction*

$$\lim_{n \rightarrow +\infty} \|u_{0,n}(\cdot + x_n) - V\|_{\dot{H}^s} = 0.$$

The second result of this paper states that the blow up rate of a minimal blow up solution can be uniformly controlled since we get a priori bound of these minimal blow up solutions.

**THEOREM 1.4** (Control of minimal blow up solutions). — *Assuming  $\mathcal{H}$ , there exists a nondecreasing function  $F_s : [0, A_s^{\sigma_s}[ \rightarrow \mathbb{R}^+$  with  $\lim_{r \rightarrow A_s^{\sigma_s}} F_s(r) = +\infty$  such that for any divergence free vector field  $u_0$  in  $\dot{H}^s$ , generating minimal blow up solution (it means  $T_*(u_0) \|u_0\|_{\dot{H}^s}^{\sigma_s} = A_s^{\sigma_s}$ ), we have the following control on the minimal blow up solution  $NS(u_0)$*

$$\forall T < T_*(u_0), \|NS(u_0)\|_{X_T^s} \leq \|u_0\|_{\dot{H}^s} F_s(T^{\frac{1}{\sigma_s}} \|u_0\|_{\dot{H}^s}).$$

**REMARK 1.2.** — Let us point out that the quantity  $T^{\frac{1}{\sigma_s}} \|u_0\|_{\dot{H}^s}$  is scaling invariant; which is obviously necessary.

The two previous theorems are the analog of results, proved in the case of the Sobolev space  $\dot{H}^{\frac{1}{2}}$ . We shall not recall all the statements existing in the literature concerning the regularity of Navier-Stokes solutions in critical spaces, such as  $\dot{H}^{\frac{1}{2}}$ . We refer for instance the reader to [7] and to the article of C. Kenig et G. Koch [13], where the authors prove that NS-solutions which remain bounded in the space  $\dot{H}^{\frac{1}{2}}$  do not become singular in finite time. Concerning Theorem 1.3, we were largely inspired by the article of W. Rusin and V. Šverák [23], in which the authors set up the key concept of minimal blow-up for data in Sobolev space  $\dot{H}^{\frac{1}{2}}$ . Firstly, they defined a critical radius  $\rho_{\frac{1}{2}}$

$$\rho_{\frac{1}{2}} = \sup\{\rho > 0 ; \|u_0\|_{\dot{H}^{\frac{1}{2}}} < \rho \implies T_*(u_0) = +\infty\}.$$

Then, they introduced a subset  $\mathcal{M}$  of  $\dot{H}^{\frac{1}{2}}$ , which describes the set of minimal-norm singularities (we speak about minimal norm in the sense of  $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$  is