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RIEMANN–ROCH POLYNOMIALS OF THE KNOWN HYPERKÄHLER MANIFOLDS

by Ángel David Ríos Ortiz

With an Appendix by Yalong Cao and Chen Jiang

Abstract. — We compute explicit formulas for the Euler characteristic of line bundles in the two exceptional examples of Hyperkähler Manifolds introduced by O'Grady. In an Appendix, Chen Jiang and Yalong Cao use our formulas to compute the Chern numbers of the example of O'Grady in dimension 10.

Résumé (*Polynômes de Riemann-Roch pour les variétés hyperkähleriennes connues*). — Nous calculons des formules explicites pour la caractéristique d'Euler de fibrés en droites pour les deux exemples exceptionnels de variétés hyperkählériennes introduits par O'Grady. Dans un appendice, Chen Jiang et Yalong Cao utilisent nos formules pour calculer le nombre de Chern de l'exemple d'O'Grady en dimension 10.

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1. Introduction

A compact Kähler manifold is called *hyperkähler* (HK) if it is simply connected and carries a holomorphic symplectic form that spans $H^{2,0}$. HK manifolds can be thought of as the higher-dimensional analogues of $K3$ surfaces, and they constitute one of the three fundamental classes of varieties with vanishing first Chern class [1].

Although any two $K3$ surfaces are deformation equivalent, this fact no longer holds in higher dimensions. The first two series of examples of deformation types in each (necessarily even) dimension were described by Beauville [1]: the first series, denoted by $K3^{[n]}$, is given by the Hilbert scheme of n points in a K3 surface. The other one is a submanifold in the Hilbert scheme of n points in an abelian surface. Generalizing the construction of a Kummer surface, this ($2n$-dimensional) deformation type is denoted by Kum_n.

Later, O'Grady introduced two new deformation types in dimensions 6 and 10 ([24],[25]), now denoted by OG6 and OG10, respectively. The construction of both exceptional examples is done by resolving a singular moduli space of sheaves on a K3 surface for OG10 and an abelian surface for OG6. In view of these analogies, it is expected that the projective geometry of HK manifolds of $K3^{[5]}$-type (respectively Kum_3-type) should be related with that of OG10-type (respectively OG6-type).

The main result of this paper (cf. Theorem 2.6) gives, for the HK manifolds described by O'Grady, closed formulas that compute the Euler characteristic of any line bundle in terms of numerical polynomials that only depend on the Beauville–Bogomolov form—a canonical quadratic form in the second cohomology group of any HK. Surprisingly the formulas turn out to be exactly the same as those of the series described by Beauville.

In order to compute these polynomials we use two different methods. The first one exploits a recent description in [15] of OG10 as a compactification of a fibration associated with a cubic 4-fold. The second one is based on the explicit descriptions of some uniruled divisors in two different models of OG6 given in [17] and [21].

Observe that in [3], the authors give a closed formula for the Riemann–Roch polynomial of OG6 in terms of the so-called λ-invariant; in our work the closed formula is obtained directly.

Finally, we would like to point out the very recent paper [14], where Chen Jiang proves the positivity of the coefficients of the Riemann–Roch polynomial for HK manifolds in general.

2. Preliminaries

Let X be an HK manifold of dimension $2n$ and q_X its Beauville–Bogomolov form [1]. Recall that the Fujiki constant c_X is defined as the rational number such that for all $\alpha \in H^2(X)$ we have the so-called Fujiki relation:

$$\int_X \alpha^{2n} = c_X q_X(\alpha)^n \tag{1}$$

Remark. — The polarized form of Fujiki's relation is

$$\int_X \alpha_1 \smile \cdots \smile \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma \in S_{2n}} q_X(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots q_X(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)}) \tag{2}$$

Huybrechts further generalized this relation to all polynomials in the Chern classes, more specifically, he proved the following:

Theorem 2.1 ([9], Corollary 23.17). — *Assume $\alpha \in H^{4j}(X, \mathbb{Q})$ is of type $(2k, 2k)$ for all small deformations of X. Then there exists a constant $C(\alpha) \in \mathbb{Q}$ such that*

$$\int_X \alpha \smile \beta^{2n-2k} = C(\alpha) \cdot q_X(\beta)^{n-k} \tag{3}$$

for all $\beta \in H^2(X, \mathbb{Q})$.

Remark. — If we set $\alpha = 1$ in Theorem 2.1 we obtain the Fujiki relation (1) and also that $a_n = c_X$.

The odd Chern classes (hence the odd Todd classes) of X vanish since the symplectic form on X induces an isomorphism between T_X and its dual. The Todd classes are topological invariants of X, so for any line bundle L in X, we combine Theorem 2.1 with Hirzebruch–Riemann–Roch theorem to get

$$\chi(X, L) = \sum_{i=0}^{n} \frac{1}{(2i)!} \int_X Td_{2n-2i}(X) \smile c_1(L)^{2i} = \sum_{i=0}^{n} \frac{a_i}{(2i)!} \cdot q_X(L)^i \tag{4}$$

where $a_i := C(Td_{2n-2i}(X))$.

Definition 2.2 (*Huybrechts, Nieper–Wißkirchen, Riess*). — The Riemann–Roch polynomial of X, denoted by $RR_X(t)$, is the polynomial

$$RR_X(t) = \sum_{i=0}^{n} \frac{a_i}{(2i)!} t^i.$$

Let us list a few well-known properties of this polynomial.

Lemma 2.3. — *Let X be an HK variety of dimension $2n$. The following properties hold:*

1. *RR_X depends only on the deformation class of X.*
2. *The constant term is $a_0 = n + 1$.*
3. *The coefficient of the highest-order term is $a_n = c_X$ and is positive.*
4. *The coefficient a_{n-1} is positive.*

Proof. — We have already observed that the Todd classes are a deformation invariant of X. Hence, each a_i (and therefore RR_X) is also a deformation invariant of X. The constant term of RR_X is the holomorphic Euler characteristic of X, this was computed [1] to be $n+1$. The constant $a_n = C(Td_0(X))$ is given by (1) so it is equal to c_X. Observe that c_X is positive because the left-hand side of the Fujiki relation (1) is a volume form.

By the first item we can assume X to be projective. Nieper [22] computed

$$\int_X c_2(X) \smile c_1(L)^{2n-2} = \binom{2n-2}{n-1} \left(\int_X c_2(X)(\sigma\overline{\sigma})^{n-1} \right) \cdot q_X(L)^{n-1}.$$

The second Todd class of X is a positive multiple of $c_2(X)$, and if L is an ample line bundle, then $q_X(L) > 0$. Therefore, a_{n-1} is positive if and only if $\int_X c_2(X)(\sigma\overline{\sigma})^{n-1}$ is positive. Fixing an HK metric compatible with the symplectic structure, the last quantity is a positive multiple of the L^2-norm of the Riemann curvature tensor (see [22]), hence positive. □

In view of the previous Lemma, we can speak of the Riemann–Roch polynomial for a deformation *type*. This has been done for the two series of examples introduced by Beauville.

Example 2.4 ([7], Lemma 5.1). — Let X be an HK of $K3^{[n]}$-type, then the Riemann–Roch polynomial is given by

$$RR_X(t) = \binom{t/2+n+1}{n}.$$

Example 2.5 ([22], Lemma 5.2). — Let X be an HK of Kum_n-type, then the Hilbert polynomial takes the form

$$RR_X(t) = (n+1)\binom{t/2+n}{n}.$$

We will say that the Riemann–Roch polynomial is of $K3^{[n]}$-type or Kum_n-type if it corresponds to one of the two examples above. Now we can state precisely the main result of this section.

Theorem 2.6. — *The Riemann–Roch polynomials for the deformation class of OG6 and OG10 are of Kum_3-type and $K3^{[5]}$-type, respectively.*

The theorem will be proved in Propositions 4.2 and 5.5 below.

3. Abelian fibered CY varieties

Let $\pi : X \to B$ be a flat surjective morphism with connected fibers between projective normal complex varieties. Denote by X_b the schematic fiber of $b \in B$. For the rest of this section, we assume that

- X has rational singularities and ω_X is trivial.
- every smooth fiber X_b is an abelian variety.

Denote by $\mathcal{O}_B(1)$ an ample line bundle on B and let $F = \pi^*(\mathcal{O}_B(1))$ be the pullback. Let L be a π-ample line bundle on X. Whenever X_b is smooth, the restriction $L_b := L|_{X_b}$ defines a polarization of the abelian variety X_b.

Recall that to any polarization on an abelian variety, one can associate a tuple of positive integers $(d_1, \dots, d_n)$ which is called the polarization type, see [12], in the following way: Since X_b is an abelian variety, we have an identification $H^2(X_b, \mathbb{Z}) \cong \bigwedge^2 H_1(X_b, \mathbb{Z})^\vee$, hence we can interpret L_b as an alternating integral form on the lattice $H_1(X_b, \mathbb{Z})$. Therefore, we can find a basis of $H_1(X_b, \mathbb{Z})$ for which L_b has the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \operatorname{diag}(d_1, \dots, d_n)$ is an integral diagonal matrix with $d_i > 0$ and $d_i | d_{i+1}$. We will denote by $(d_1, \dots, d_n)$ the type of L_b. Since the morphism is flat, the type remains constant on the smooth locus of π. The following is a generalization of [29, Claim 12]:

Theorem 3.1. — *Let L be a π-ample line bundle on X and let $(d_1, \dots, d_n)$ be the type of L_b for a smooth fiber X_b. Then for any $m \in \mathbb{Z}$, the sheaf $\pi_*(L \otimes F^{\otimes m})$ is locally free of rank $d_1 \cdots d_n$ and all higher direct images vanish. Moreover,*

$$h^p(X_b, (L \otimes F^{\otimes m})|_{X_b}) = \begin{cases} d_1 \cdots d_n & p = 0, \\ 0 & p > 0. \end{cases}$$

Proof. — Let $k > 0$ be an integer such that $M = L \otimes F^{\otimes k}$ is ample. Let X_b be a smooth fiber, and denote by M_b the restriction of M. Then

$$h^p(X_b, M_b) = h^p(X_b, L_b) = \begin{cases} d_1 \cdots d_n & p = 0, \\ 0 & p > 0. \end{cases}$$

Therefore, the higher direct image sheaves $R^p\pi_*M$ are torsion for $p > 0$. Let $\epsilon : \widetilde{X} \to X$ be a resolution of singularities of X. Since X has rational singularities, $\epsilon_*(\omega_{\widetilde{X}}) = \omega_X = \mathcal{O}_X$ and $R^q\epsilon_*\omega_{\widetilde{X}} = 0$ for every $q > 0$. Therefore, the Grothendieck spectral sequence

$$R^p\pi_*(R^q\epsilon_*\omega_{\widetilde{X}} \otimes M) \implies R^{p+q}(\epsilon \circ \pi)_*(\omega_{\widetilde{X}} \otimes \epsilon^*M)$$

degenerates and so $R^p(\epsilon \circ \pi)_*(\omega_{\widetilde{X}} \otimes \epsilon^*M) \cong R^p\pi_*(M)$. On the other hand, the divisor $\epsilon^*(M)$ is big and nef, so Theorem 2.2. in [10] states that $R^p(\epsilon \circ \pi)_*(\omega_{\widetilde{X}} \otimes \epsilon^*M)$ is torsion free for $p \geq 0$. We conclude that $R^p\pi_*(M)$ must vanish for $p > 0$.

Theorem 12.11 of [11] states that if $H^p(X_b, M_b)$ vanishes for all $b \in B$, then the natural map

$$R^{p-1}\pi_*M \otimes_{\mathcal{O}_b} k(b) \to H^{p-1}(X_b, M_b)$$