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CONVEX COMPACT SURFACES WITH NO BOUND ON THEIR SYNTHETIC RICCI CURVATURE

by Constantin Vernicos

Abstract. — Using refraction in the setting of normed vector spaces allows us to present an example of a convex compact surface which admits no lower bound on its Ricci curvature as defined by Lott–Villani and Sturm.

Résumé (*Surfaces convexes et compactes n'admettant pas de borne de leur courbure de Ricci synthétiques*). — L'utilisation de la notion de réfraction dans le cadre des espaces vectoriels normés permet de construite un exemple de surface convexe et compacte qui n'est pas de courbure de Ricci minorée telle que défini par Lott-Villani et Sturm.

Introduction and statement of results

Many notions of curvature bounds adapted to a metric measure space have been defined to extend the ones existing in Riemannian geometry. Most of them heavily rely on comparison to the Euclidean space and that is why they are quite restrictive. For instance, a normed vector space is CAT(0) if and only if it is an Euclidean space; as a consequence, the only Finsler spaces which can be CAT(0) are Riemannian (see also [2]). The same thing happens with the

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Alexandroff spaces. It is even more general in that case, for an Alexandroff metric space happens to be almost Riemannian manifold (see [1] for a precise statement).

Some older notions, such as the Busemann convexity, are less restrictive. However, they might not pass to the Gromov–Hausdorff limit of a sequence of metric measured spaces; for instance, this happens when one approximates a non-strictly convex norm by strictly convex ones. The family of strictly convex normed spaces obtained are Busemann convex and converge to the non-strictly convex ones which are not. In the light of the current interest in understanding the limit spaces arising as limits of Riemannian metric space, with Ricci curvature bounded from below, for instance, this is a huge flaw.

Following the work of Lott & Villani [6] and Sturm [15, 16], a new family of notions of curvature bounded spaces arose. They involve the convexity of an operator on the L^2-Wasserstein space, which is a metrization of the space of probability measures with finite 2-th moment. Among them one finds the spaces satisfying the curvature dimension condition $CD(K,N)$ or the measure contraction property $MCP(K,N)$. The latter may be seen as a measure analog to the Busemann convexity, the former as a generalization of having Ricci curvature bounded from below by K and being of dimension less than N. We will refer to this last notion as synthetic Ricci curvature and describe such spaces as admitting a lower bound on their synthetic Ricci curvature. An example is given by a normed vector space of dimension n which satisfies the curvature dimension condition $CD(0,n)$ (see [17] in the Appendix).

Another point of view on curvature in metric spaces is based on analytical inequalities. For instance, Cordero-Erausquin, McCann, and Schmuckenschläger [3] looked at the Brascamp–Lieb inequality which is a generalization of the Prekopa–Leindler inequality that can be used to prove the Brunn–Minkoswki inequality in the Euclidean space.

The interesting aspect on which this paper is based is that most notions of curvature deriving from the work of Lott–Villani and Sturm imply a Brunn–Minkowski inequality, hence our focus on this inequality (see also [8, 9] for a recent study on the relation between the Brunn–Minkowski and the CD condition).

Our main result is the following:

THEOREM 1. — *There exists a compact $C^{1,1}$ convex surface in R^3 with the norm $\|(x,y,z)\| = \sqrt{x^2+y^2}+|z|$ which admits no lower bounds on its synthetic Ricci curvature.*

The idea of that example came from the study of reflections and refraction in normed (not necessarily reflexive) vector spaces. Section 2 focuses on a specific example which allows us to obtain our convex set in Section 3.

It is worth mentioning here that the specific example of Section 2 also shows that the CD property is not preserved by gluing two CD spaces along their isometric boundaries. This behavior distinguishes the CD property from other properties, such as Alexandroff spaces (see [13, 5]).

The main reason why the example in Section 2 is not a CD space is due to the particular structure of geodesics which branch along a hyperplane. It is known that such branching does not go along with the CD property unless one has a particular measure and metric structure (see [7], pointed out to us by an anonymous referee as this paper was not available when the present work was done).

Section 3 is a perturbation of Section 2's example which smooths the space a bit and probably gets rid of the branching, but without allowing a synthetic curvature lower bound. One must also emphasize here that if the smoothing were C^2 then a lower bound would exist. Hence the nonexistence is not an immediate thing.

1. Definitions and notations

A metric measured space (X, d, μ) is a space X endowed with a distance d and a measure μ, usually a Borel one. Let us fix a metric measured space. For any pairs of point m_0, $m_1 \in X$, we call $m_s \in X$ an s-intermediate point from m_0 to m_1 if and only if

$$d(m_0, m_s) = sd(m_0, m_1) \quad \text{and} \quad d(m_s, m_1) = (1-s)d(m_0, m_1).$$

Let K_0 and K_1 be two compact sets in X, the set of s-intermediate points from points of K_0 to points of K_1 will be denoted by

$$M_s(K_0, K_1).$$

If $M_s(K_0, K_1)$ is not measurable, we will still denote its outer measure by

$$\mu\big(M_s(K_0, K_1)\big).$$

Let us first start with the classical Brunn–Minkowski inequality:

Definition 1.1 (Classical Brunn–Minkowski inequality). — Let N be greater than 1. We say that the Brunn–Minkowski inequality $BM(0, N)$ holds in the metric measured space (X, d, μ) if for every pair of compact sets K_0 and K_1, the following inequality is satisfied:

$$\mu^{1/N}\big(M_s(K_0, K_1)\big) \geq (1-s)\mu^{1/N}(K_0) + s\mu^{1/N}(K_1). \tag{1}$$

We also say that $BM(0, +\infty)$ holds if and only if

$$\mu\big(M_s(K_0, K_1)\big) \geq \mu^{1-s}(K_0)\mu^s(K_1). \tag{2}$$

REMARK 1.2. — Notice that if for some $n \in R^*$, and t,a and $b \in \mathbb{R}$, the inequality $t \geq (sa^{1/n} + (1-s)b^{1/n})^n$ holds, then from the concavity of the logarithm we have

$$\begin{aligned}\ln t &\geq n \ln\big(sa^{1/n} + (1-s)b^{1/n}\big)\\ &\geq s\ln a + (1-s)\ln b.\end{aligned}$$

Hence, any $BM(0,N)$ implies $BM(0,\infty)$.

Now the general Brunn–Minkowski inequality $BM(K,N)$ requires the introduction of a family of functions depending on K, N, and $s \in [0,1]$ denoted by $\tau_{K,N}^{(s)} \colon \mathbb{R}^+ \to \mathbb{R}^+$. For a fixed $s \in [0,1]$ and $\theta \in \mathbb{R}^+$, $\tau_{K,N}^{(s)}(\theta)$ is continuous, nonincreasing in N, and nondecreasing in K. Its exact definition is not important for our applications, refer to [16].

DEFINITION 1.3 (Generalized Brunn–Minkowski inequality). — Let N be greater than 1 and $K \in \mathbb{R}$. We say that the Brunn–Minkowski inequality $BM(K,N)$ holds in the metric measured space (X,d,μ) if for every pair of compact set K_0 and K_1, the following inequality is satisfied:

$$\mu^{1/N}\big(M_s(K_0,K_1)\big) \geq \tau_{K,N}^{(1-s)}(\vartheta)\mu^{1/N}(K_0) + \tau_{K,N}^{(s)}(\vartheta)\mu^{1/N}(K_1). \tag{3}$$

where ϑ is the minimal (respectively maximal) length of a geodesic between a point in K_0 and a point in K_1 if $K \geq 0$ (respectively $K < 0$).

We can also define the $BM(K,+\infty)$ as follows:

$$\mu\big(M_s(K_0,K_1)\big) \geq \mu^{1-s}(K_0)\mu^s(K_1)e^{Ks(1-s)\vartheta^2/2}. \tag{4}$$

The curvature dimension property, denoted by $CD(K,N)$, is a generalization of the following sentence on metric measures spaces:

> The space has dimension less than N and the Ricci curvature is bigger than K.

It is defined in terms of a convexity property of the entropy along geodesics in the space of probability of the metric space (see [16] for more precise statements).

For our purpose we only need to know the following properties of a space satisfying a curvature dimension property (see K.T. Sturm [16]).

PROPERTY 1.4. — *Let (X,d,μ) be a metric measured space, $K \in \mathbb{R}$. The following implications are valid:*

1. *Suppose $CD(K,N)$ holds. If $K' \leq K$, then $CD(K',N)$ holds as well. If $N' > N$, then $CD(K,N')$ holds as well.*
2. *Suppose $CD(K,N)$ holds. Then for any $\alpha,\beta > 0$, the metric measured space $(X,\alpha d,\beta\mu)$ satisfies the $CD(K/\alpha^2,N)$ condition.*

3. *$CD(0,N)$ implies $BM(0,N)$ and, more generally, $CD(K,N)$ implies $BM(K,N)$.*
4. *$CD(K,N)$ implies the Bishop–Gromov volume growth inequality with the Riemannian space of constant curvature K and dimension N.*

2. Brunn–Minkowski inequality is not preserved in a two-layer Banach space

In this section we are going to consider the vector space $\mathbb{R}^2$ and the hyperplane $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$. We are going to put the classical Euclidean ℓ^2 norm $\|(x,y)\|_2 = \sqrt{x^2+y^2}$ on the half-space $y > 0$ and the ℓ^1 norm $\|(x,y)\|_1 = |x| + |y|$ on the half-space $y < 0$. Given $P = (x,y)$ and $Q = (x',y')$ in $\mathbb{R}^2$ we define the distance $d_{2,1}$ by

$$\begin{cases} \text{if } y > 0, y' > 0 & d_{2,1}(P,Q) = \|P-Q\|_2 \\ \text{if } y < 0, y' < 0 & d_{2,1}(P,Q) = \|P-Q\|_1 \\ \text{if } y > 0 \text{ and } y' < 0 & d_{2,1}(P,Q) = \inf_{Z\in\mathcal{H}} \|Z-P\|_2 + \|Q-Z\|_1 \end{cases}$$

This is actually the length distance when curves on the upper half-plane are measured thanks to their Euclidean length, and on the lower half-plane thanks to their ℓ^1-length. It is important here that the restriction of both norms coincides on the hyperplane $\mathcal{H}$.

Now let us specify the measures $\mathfrak{m}$ we will use here. In Finsler geometry there is no canonical measure as in Riemannian geometry. One has to choose a consistent normalization of the Lebesgue measure on each tangent space (see [1]). One possibility is to fix the volume of each tangent ball equal to π, this gives the so-called Busemann volume. In our case, if we denote by λ the standard Lebesgue measure, that is such that π is the measure of the standard Euclidean disk, then on the lower half-space our measure would be $\alpha\lambda$ with $\alpha = \pi/2$. Other normalization exists (see again [1]).

We shall denote by $(\mathbb{R}^2, d_{2,1}, \mathfrak{m})$ the metric measured space obtained this way.

PROPERTIES 2.1. — *Let $\alpha \in \mathbb{R}$ and $X_0 = (\rho,\theta)$ be in the upper half-plane in polar coordinates centered at the point $O_\alpha = (\alpha, 0)$. Consider $X_1 = (\alpha, y)$ be in the lower half-plane in Cartesian coordinates ($y < 0$), then*

- *the geodesic joining X_0 to X_1 is composed of the line segment from X_0 to the point O_α and from the point O_α to X_1. It is unique;*
- *the distance between X_0 to X_1 is equal to $\rho - y$;*
- *let X_s be the s-intermediate point between X_0 to X_1,*
  1. *if $s(\rho - y) < \rho$, then X_s belongs to the upper half-plane and lies on the affine segment from X_0 to the point O_α, and $X_s = \big((1-s)\rho + sy, \theta\big)$ in polar coordinates;*