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LIMIT THEOREMS FOR HOROCYCLE FLOWS

BY Alexander BUFETOV AND Giovanni FORNI

Abstract. – The main results of this paper are limit theorems for horocycle flows on compact surfaces of constant negative curvature.

One of the main objects of the paper is a special family of horocycle-invariant finitely additive Hölder measures on rectifiable arcs. An asymptotic formula for ergodic integrals for horocycle flows is obtained in terms of the finitely-additive measures, and limit theorems follow as a corollary of the asymptotic formula.

The objects and results of this paper are similar to those in [15, 16], [5] and [6] for translation flows on flat surfaces. The arguments are based on the representation theory methods developed in [12] for the classification of invariant distributions, the solution of the cohomological equation and the asymptotics of ergodic averages of horocycle flows.

Résumé. – Le but de cet article est de démontrer des théorèmes limites pour les flots horocycliques sur les surfaces compactes à courbure négative constante. Notre principal outil est une famille particulière de mesures additives, höldériennes, invariantes par le flot horocyclique sur les arcs rectifiables. Une formule asymptotique pour les intégrales ergodiques des flots horocycliques est ensuite établie en termes des mesures additives, et les théorèmes limites en découlent. Les constructions et les résultats ici développés ressemblent à ceux de [15], [16], [5] et [6] qui traitent des flots de translation sur les surfaces plates. Les preuves sont basées sur des méthodes de la théorie des réprésentations développées en [12] pour l'étude de l'équation cohomologique et du comportement asymptotique des moyennes ergodiques des flots horocycliques.

1. Introduction

1.1. Outline of the main results

The aim of this paper is to obtain limit theorems for horocycle flows on compact surfaces of constant negative curvature.

Our limit theorems admit the simplest formulation in the case when the smallest positive eigenvalue μ_0 of the Laplace operator on the surface of curvature -1 is strictly less than

$1/4$ (equivalently, when the spectral decomposition of the space of square-integrable functions on our surface into irreducible unitary representations of the modular group contains representations of the complementary series).

In this case, the variance of the ergodic integrals (up to time $T > 0$) of a generic smooth function grows at the rate $T^{\frac{1+\nu_0}{2}}$, where $\nu_0 := \sqrt{1-4\mu_0}$, and its ergodic integrals, normalized to have variance 1, converge in distribution to a nondegenerate compactly supported measure on the real line.

The situation is more complicated for surfaces whose spectral decomposition only contains representations of the principal series (or more generally for functions supported on irreducible representations of the principal series).

In this case, the variance of ergodic integrals (up to time $T > 0$) of any smooth function which is not a coboundary grows at the rate $T^{\frac{1}{2}}$, but its ergodic integrals, normalized to have variance 1, converge in distribution to an orbit of an infinite-dimensional *quasi-periodic* flow in the space of random variables with compactly supported distributions. The frequencies of this quasi-periodic motion are determined by the eigenvalues larger than $1/4$ of the Laplace-Beltrami operator on the hyperbolic surface. We are not able to determine whether the limit distribution exists in this case; we conjecture that it does not. In fact, the limit distribution will exist for all smooth functions which are not coboundaries if and only if all random variables in each of the invariant subtori of our infinite dimensional torus have exactly the same probability distribution (see §5.4).

Our argument relies on the classification, due to Flaminio and Forni [12], of distributions (in the sense of S. L. Sobolev and L. Schwartz(1)) invariant under a given horocycle flow. One of the main objects of our paper is a closely related space of finitely-additive Hölder measures on rectifiable arcs on our surface, invariant under the complementary horocycle flow. We classify these measures and establish an explicit bijective correspondence between them and the subspace of the Flaminio-Forni space given by invariant distributions corresponding to positive eigenvalues of the Casimir operator. This isomorphism yields a natural duality between the spaces of invariant distributions for the two horocycle flows on a surface. We further establish an asymptotic formula for ergodic integrals in terms of the finitely-additive measures. Our limit theorems are obtained as corollaries of the asymptotic formula. Informally, the limit theorems claim that the normalized ergodic integrals of horocycle flows converge in distribution to the probability distributions of finitely-additive measures of horocycle arcs.

The objects and results of this paper are similar to those in [15, 16] and especially [5, 6], [4] for translation flows on flat surfaces. The methods here are completely different, however, and are based on those in [12].

The remainder of this section is organized as follows. In §1.2 we make some brief historical remarks. In §1.3 we establish our notation and recall the main properties of invariant distributions and basic currents for horocycle flows. In §1.4 we state our main theorems on finitely additive Hölder measures on rectifiable arcs, invariant with respect to the unstable

(1) The term "distribution" is used in two very different senses in our paper: first, probability distributions of random variables and, second, distributions of Sobolev and Schwartz. For instance, "limit distributions" refer to the first meaning, while "invariant distributions" to the second. We hope that our precise meaning is always clear from the context.

(stable) horocycle (Theorem 1.1) and on the related additive cocycle for the stable (unstable) horocycle (Theorem 1.2). We also state several important corollaries of the above mentioned theorems (Corollary 1.1 and Corollary 1.2). In §1.5 we state our results on the asymptotics of ergodic integrals, in particular we state an approximation theorem for ergodic integrals of sufficiently smooth zero-average functions in terms of additive cocycles (Theorem 1.3) and our results on limit distributions of normalized ergodic integrals (Theorem 1.4 and Theorem 1.5). We then state our conditional results about the existence of limit distributions for functions supported on irreducible components of the principal series (Corollary 1.4 and Corollary 1.5). In §1.6 we introduce currents of dimension 2 (and degree 1) associated to our finitely additive measures on rectifiable arcs. We then state a duality theorem which affirms that such currents can be written in terms of invariant distributions for the unstable (stable) horocycle flow (Theorem 1.7). The duality theorem leads to a complete classification of the class of finitely additive Hölder measures axiomatically defined in §1.4 (see Definition 1.2), in the sense that our construction gives the space of all finitely additive Hölder measures with the listed properties (Theorem 1.8). It also allows us to establish a direct relation between the lifts of our additive cocycles to the universal cover and Γ-conformal invariant distributions on the boundary of the Poincaré disk (Theorem 1.9).

1.2. Historical remarks

The classical horocycle flow on a compact surface of constant negative curvature is a main example of a unipotent, parabolic flow. Its ergodic theory has been extensively studied. It is known that the flow is minimal [21], uniquely ergodic [17], has Lebesgue spectrum and is therefore strongly mixing [29], in fact mixing of all orders [27], and has zero entropy [20]. Its finer ergodic and rigidity properties, as well as the rate of mixing, were investigated by M. Ratner in a series of papers [30, 31], [32, 33]. In joint work with L. Flaminio [12], the second author has proved precise bounds on ergodic integrals of smooth functions. Those bounds already imply, as proved in [12], that all weak limits of probability distributions of normalized ergodic integrals of generic smooth functions have (non degenerate) compact support.

In the case of finite-volume surfaces, the classification of invariant measures is due to Dani [10]. The asymptotic behaviour of averages along closed horocycles in the finite-volume case has been studied by D. Zagier [36], P. Sarnak [34], D. Hejal [22] and more recently in [12] and by A. Strömbergsson [35]. The flows on general geometrically finite surfaces have been studied by M. Burger [7].

Invariant distributions, and, more generally, eigendistributions for smooth dynamical systems were already considered in 1955 by S. V. Fomin [14], who constructed a full system of eigendistributions for a linear toral automorphism.

In the case of horospherical foliations of symmetric spaces $X = G/K$ of non-compact type of connected semi-simple Lie groups G with finite center, invariant distributions are related to *conical distributions* on the space of horocycles introduced in the work of S. Helgason [23].

Invariant distributions for the horocycle flow appear in the asymptotics for the equidistribution of long closed horocycle on finite-volume non-compact hyperbolic surfaces in work of P. Sarnak [34]. To the authors' best knowledge this is the first appearance of invariant distributions in the context of quantitative equidistribution. Sarnak's work was later generalized to arbitrary horocycle arcs and to the horocyle flow also in the compact case in [12] (see also [22], [35]).

Other similar (parabolic, uniquely ergodic) systems for which the asymptotics and the limit distributions of ergodic integrals have been studied include translation flows on surfaces of higher genus and interval exchange transformations, substitution dynamical systems and Vershik's automorphisms, and nilflows on homogeneous spaces of the Heisenberg group. The latter are related to the asymptotic behavior of theta sums. For translation flows and interval exchange transformations, results on the growth of ergodic integrals were proved conditionally in the work of A. Zorich [37, 38], [39] and M. Kontsevich [26] and later fully proved in [16] and by A. Avila and M. Viana [1]. An asymptotic formula for ergodic integrals and limit theorems for translation flows were obtained in [5], [6] and [4]. Similar results for suspension flows over Vershik's automorphisms were obtained in [5]. Limit theorems for theta sums were proved by W. B. Jurkat and J. W. Van Horne [24, 25], by J. Marklof [28] and more recently in stronger form by F. Cellarosi [8]. Invariant distributions and asymptotics of ergodic integrals for Heisenberg nilflows were studied in [13], which generalizes the asymptotics for theta sums proved by H. Fiedler, W. B. Jurkat and O. Körner [11] .

1.3. Definitions and notation

Let Γ be a co-compact lattice in $PSL(2,\mathbb{R})$ and let $M := \Gamma\backslash D$ be the corresponding hyperbolic surface obtained as a quotient of the Poincaré disk D under standard action of Γ by linear fractional transformations. Since $PSL(2,\mathbb{R})$ acts freely and transitively on the unit tangent bundle of the Poincaré disk, the unit tangent bundle SM of the hyperbolic surface M can be identified with the homogeneous space $\Gamma\backslash PSL(2,\mathbb{R})$. Let $\{X, U, V\}$ be the basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $PSL(2,\mathbb{R})$ given by the infinitesimal generators of the geodesic flow and of the stable and unstable horocycle flows. The following commutation relations hold:

$$[X,U] = U\,, \quad [X,V] = -V\,, \quad [U,V] = 2X\,. \tag{1.1}$$

Let $\{\hat X, \hat U, \hat V\}$ be the frame of the cotangent bundle dual to the frame $\{X, U, V\}$ of the tangent bundle, that is,

$$\begin{aligned} \hat X(X) &= 1\,, \quad \hat X(U) = 0 \quad \hat X(V) = 0\,; \\ \hat U(X) &= 0\,, \quad \hat U(U) = 1 \quad \hat U(V) = 0\,; \\ \hat V(X) &= 0\,, \quad \hat V(U) = 0 \quad \hat V(V) = 1\,. \end{aligned}$$

Let $|\hat X|$, $|\hat U|$ and $|\hat V|$ denote the 1-dimensional measures on SM transverse to the 2-dimensional foliations $\{\hat X = 0\}$, $\{\hat U = 0\}$ and $\{\hat V = 0\}$ given respectively by the 1-forms $\hat X$, $\hat U$ and $\hat V$. In other terms, if γ is any rectifiable path transverse to the foliation $\{\hat X = 0\}$, ($\{\hat U = 0\}$, $\{\hat V = 0\}$), then respectively,

$$\int_\gamma |\hat X| = \left|\int_\gamma \hat X\right| \quad \left(\int_\gamma |\hat U| = \left|\int_\gamma \hat U\right|\,, \quad \int_\gamma |\hat V| = \left|\int_\gamma \hat V\right|\right).$$