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SCATTERING RESONANCES FOR HIGHLY OSCILLATORY POTENTIALS

BY Alexis DROUOT

Abstract. – We study resonances of compactly supported potentials $V_\varepsilon(x) = W(x, x/\varepsilon)$ where $W : \mathbb{R}^d \times \mathbb{R}^d/(2\pi\mathbb{Z})^d \to \mathbb{C}$, d odd. That means that V_ε is a sum of a slowly varying potential, W_0, and one oscillating at frequency $1/\varepsilon$. When $W_0 \equiv 0$ we prove that there are no resonances above the line $\operatorname{Im}\lambda = -A\ln(\varepsilon^{-1})$, except a simple resonance near 0 when $d = 1$. We show that this result is optimal by constructing a one-dimensional example. This settles a conjecture of Duchêne-Vukićević-Weinstein [12]. When $W_0 \neq 0$ and W smooth we prove that resonances in fixed strips admit an expansion in powers of ε. The argument provides a method for computing the coefficients of the expansion. We produce an effective potential converging uniformly to W_0 as $\varepsilon \to 0$ and whose resonances approach resonances of V_ε modulo $O(\varepsilon^4)$. This improves the one-dimensional result of Duchêne, Vukićević and Weinstein and extends it to all odd dimensions.

Résumé. – Nous étudions les résonances de potentiels à support compact $V_\varepsilon(x) = W(x, x/\varepsilon)$, où $W : \mathbb{R}^d \times \mathbb{R}^d/(2\pi\mathbb{Z})^d \to \mathbb{C}$ et d est impair. Ainsi, V_ε est la somme d'un potentiel qui varie lentement W_0 et d'un potentiel qui oscille à fréquence $1/\varepsilon$. Quand $W_0 \equiv 0$ nous prouvons que V_ε n'a pas de résonances dans la zone $\{\operatorname{Im}\lambda \geq -A\ln(\varepsilon^{-1})\}$ mise à part une unique résonance proche de 0 si $d = 1$. Nous montrons par un exemple explicite que ce résultat est optimal. Cela prouve une conjecture de Duchêne-Vukićević-Weinstein [12]. Quand $W_0 \neq 0$ et W est lisse nous montrons que les resonances de V_ε qui restent bornées lorsque ε tend vers 0 admettent une expansion en puissances de ε. Les arguments de la preuve permettent de calculer les coefficients de cette expansion. Nous construisons un potentiel effectif qui converge uniformément vers W_0 lorsque ε tend vers 0 et dont les résonances sont à distance $O(\varepsilon^4)$ de celles de W_0. Cela améliore et étend les résultats de Duchêne, Vukićević et Weinstein à toutes les dimensions impaires.

1. Introduction

In this paper we are interested in the poles of the meromorphic continuation of $(-\Delta + \mathcal{V} - \lambda^2)^{-1}$ where d is odd and $\mathcal{V} : \mathbb{R}^d \to \mathbb{C}$ is a bounded compactly supported potential. These poles called scattering resonances appear in many physical situations, for instance their imaginary parts are the rates of decay of waves scattered by $\mathcal{V}$.

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Let $-\Delta \geq 0$ be the free Laplacian on $\mathbb{R}^d$. The operator $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, well defined as an operator $L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$ for $\operatorname{Im}\lambda > 0$, extends to a meromorphic family of bounded operators $L^2_{\text{comp}}(\mathbb{R}^d) \to H^2_{\text{loc}}(\mathbb{R}^d)$ for $\lambda \in \mathbb{C}$ (see §1.5 for review of notation). This family admits one simple pole at 0 if $d = 1$ and is entire if $d \geq 3$. If $\mathcal{V}$ is a bounded compactly supported function on $\mathbb{R}^d$ then $R_{\mathcal{V}}(\lambda) = (-\Delta + \mathcal{V} - \lambda^2)^{-1}$ is well defined for $\operatorname{Im}\lambda \gg 1$ as an operator $L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$. It extends to a meromorphic family of operators $L^2_{\text{comp}}(\mathbb{R}^d) \to H^2_{\text{loc}}(\mathbb{R}^d)$. In this sense, the resonances of a real-valued potential $\mathcal{V}$—similarly, the poles of the meromorphic continuation of $R_{\mathcal{V}}(\lambda)$—are a generalization of eigenvalues of $-\Delta + \mathcal{V}$: each eigenvalue E of $-\Delta + \mathcal{V}$ is negative and generates a resonance $i\sqrt{-E}$, and conversely every resonance λ of $\mathcal{V}$ in the upper half-plane lies in $i[0,\infty)$ and corresponds to the eigenvalue λ^2. Resonances of $\mathcal{V}$ in the lower half-plane are not related to eigenvalues of $-\Delta + \mathcal{V}$, though they quantize the rate of decay of waves scattered by $\mathcal{V}$. We refer to [15, §2, 3] for a complete introduction to resonances in potential scattering.

Let W be a *bounded* complex valued function with support in $\mathbb{B}^d(0,L) \times \mathbb{T}^d$. We define V_ε as

$$V_\varepsilon(x) = W\left(x, \frac{x}{\varepsilon}\right).$$

If W is formally given by

$$W(x,y) = \sum_{k\in\mathbb{Z}^d} W_k(x) e^{iky}$$

we can write V_ε as a highly oscillatory perturbation of W_0:

$$V_\varepsilon(x) = W_0(x) + V_\sharp(x), \quad V_\sharp(x) = \sum_{k\neq 0} W_k(x) e^{ikx/\varepsilon}. \tag{1.1}$$

In this paper we study resonances of potentials V_ε given by (1.1).

1.1. Main results

The first theorem concerns the case of a vanishing slowly varying part. In the notations of (1.1) we will assume for this result that $W \in L^\infty_0(\mathbb{B}^d(0,L) \times \mathbb{T}^d)$ (i.e., $\operatorname{supp}(W)$ is a compact subset of $\mathbb{B}^d(0,L) \times \mathbb{T}^d$ and W is uniformly bounded) and that moreover,

$$\begin{aligned} \exists s \in (0,1), \quad & \sum_{k\neq 0} \frac{|W_k|_{H^s}}{|k|^s} < \infty \text{ if } d = 1, \\ & \sum_{k\neq 0} \frac{\|W_k\|_1}{|k|} < \infty \text{ if } d \geq 3. \end{aligned} \tag{1.2}$$

THEOREM 1. – *Let W be in $L^\infty_0(\mathbb{B}^d(0,L) \times \mathbb{T}^d, \mathbb{C})$ such that $W_0 \equiv 0$ and (1.2) holds. Then there exists C, c, A three positive constants such that*

$$\begin{aligned} &\textit{if } d = 1, \ \operatorname{Res}(V_\varepsilon) \setminus \mathbb{D}\left(0, c\varepsilon^{s/2}\right) \subset \left\{\lambda \in \mathbb{C} : \ \operatorname{Im}\lambda \leq C - A\ln(\varepsilon^{-1})\right\}; \\ &\textit{if } d \geq 3, \ \operatorname{Res}(V_\varepsilon) \subset \left\{\lambda \in \mathbb{C} : \ \operatorname{Im}\lambda \leq C - A\ln(\varepsilon^{-1})\right\}. \end{aligned}$$

This settles a conjecture of [12]: for odd dimensions $d \geq 3$ and ε small enough the potential V_ε does not have a bound state. In §2.3 we construct a step-like function W such that $V_{\pi/(2n)}$ has a resonance $\lambda_n \sim -i\ln(n)$ as $n \to +\infty$. This shows that one cannot improve the rate of escape of resonances given by Theorem 1 in dimension 1.

In the next statements we always assume that W is smooth. We consider the case $W_0 \neq 0$. If λ_0 is a simple resonance of W_0 we can write

$$R_{W_0}(\lambda) = \frac{iu \otimes v}{\lambda - \lambda_0} + H(\lambda),\ \ H(\lambda) \text{ holomorphic near } \lambda_0, \tag{1.3}$$

for some functions $u, v \in H^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$ called resonant states. As the potential V_ε given by (1.1) converges weakly to W_0, it is natural to expect that resonances of V_ε converge to resonances of W_0. In fact a much stronger statement holds:

THEOREM 2. – *Let W belong to $C_0^\infty(\mathbb{B}^d(0,L)\times\mathbb{T}^d, \mathbb{C})$ and V_ε be given by* (1.1)*. Let λ_0 be a simple resonance of W_0. In a neighborhood of λ_0 and for ε small enough the potential V_ε admits a unique resonance λ_ε. Moreover, for any N,*

$$\lambda_\varepsilon = \lambda_0 + c_2\varepsilon^2 + c_3\varepsilon^3 + \cdots + c_{N-1}\varepsilon^{N-1} + O(\varepsilon^N),\ \ c_j \in \mathbb{C}.$$

If u, v are the resonant states of (1.3) *then*

$$\begin{aligned} c_2 &= i\int_{\mathbb{R}^d} \Lambda_0(x)u(x)v(x)dx, & c_3 &= i\int_{\mathbb{R}^d} \Lambda_1(x)u(x)v(x)dx, \\ \Lambda_0 &= \sum_{k\neq 0} \frac{W_k W_{-k}}{|k|^2}, & \Lambda_1 &= -2\sum_{k\neq 0} \frac{W_{-k}((k\cdot D)W_k)}{|k|^4}. \end{aligned} \tag{1.4}$$

If W is real-valued then so are Λ_0 and Λ_1. In §3.1 we will prove a version of Theorem 2 for resonances of higher multiplicity. Theorem 2 implies that perturbations of W_0 by a high frequency potential $V_\sharp$ enjoy some similarities with suitable analytic perturbations of W_0. In fact we have the following

THEOREM 3. – *Assume that W belongs to $C_0^\infty(\mathbb{B}^d(0,L)\times\mathbb{T}^d, \mathbb{C})$ and that V_ε is given by* (1.1)*. Let $V_{\mathrm{eff},\varepsilon} = W_0 - \varepsilon^2\Lambda_0 - \varepsilon^3\Lambda_1$ where Λ_0, Λ_1 are given in* (1.4)*. For every bounded family $\varepsilon \mapsto \mu_\varepsilon$ of simple resonances of $V_{\mathrm{eff},\varepsilon}$ there exists a family of resonances $\varepsilon \mapsto \lambda_\varepsilon$ of V_ε such that*

$$|\lambda_\varepsilon - \mu_\varepsilon| = O(\varepsilon^4).$$

Conversely for every bounded family $\varepsilon \mapsto \lambda_\varepsilon$ of simple resonances of V_ε there exists a family of resonances $\varepsilon \mapsto \mu_\varepsilon$ of $V_{\mathrm{eff},\varepsilon}$ such that

$$|\lambda_\varepsilon - \mu_\varepsilon| = O(\varepsilon^4).$$

The potential $V_{\mathrm{eff},\varepsilon}$ plays the role of an effective potential. In dimension one Λ_0 was already derived in [12].

We next give a uniform description of the behavior of resonances of V_ε as $\varepsilon \to 0$. For $W_0 \in C_0^\infty(\mathbb{B}^d(0,L), \mathbb{C})$ we define $m_{W_0}(\lambda_0)$ the multiplicity of a resonance λ_0 of W_0. If