ISSN 0012-9593

quatrième série - tome 52 fascicule 4 juillet-août 2019

ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 52, 2019, p. 927 à 1023

CUTTING SEQUENCES ON BOUW-MÖLLER SURFACES: AN $\mathcal{S}$-ADIC CHARACTERIZATION

BY DIANA DAVIS, IRENE PASQUINELLI AND CORINNA ULCIGRAI

ABSTRACT. – We consider a symbolic coding for geodesics on the family of *Veech surfaces* (translation surfaces rich with affine symmetries) recently discovered by *Bouw* and *Möller*. These surfaces, as noticed by *Hooper*, can be realized by cutting and pasting a collection of *semi-regular polygons*. We characterize the set of symbolic sequences (*cutting sequences*) that arise by coding linear trajectories by the sequence of polygon sides crossed. We provide a full characterization for the closure of the set of cutting sequences, in the spirit of the classical characterization of Sturmian sequences and the recent characterization of Smillie-Ulcigrai of cutting sequences of linear trajectories on regular polygons. The characterization is in terms of a system of finitely many substitutions (also known as an *$\mathcal{S}$-adic presentation*), governed by a one-dimensional continued fraction-like map. As in the Sturmian and regular polygon case, the characterization is based on *renormalization* and the definition of a suitable combinatorial *derivation* operator. One of the novelties is that derivation is done in two steps, without directly using Veech group elements, but by exploiting an affine diffeomorphism that maps a Bouw-Möller surface to the *dual* Bouw-Möller surface in the same Teichmüller disk. As a technical tool, we crucially exploit the presentation of Bouw-Möller surfaces via *Hooper diagrams*.

RÉSUMÉ. – On considère un codage symbolique des géodésiques sur une famille de surfaces de Veech (surfaces de translation riches en symétries affines) récemment découverte par Bouw et Möller. Ces surfaces, comme l'a remarqué Hooper, peuvent être réalisées en coupant et collant une collection de polygones semi-réguliers. Dans cet article, on caractérise l'ensemble des suites symboliques (« suites de coupage ») qui correspondent au codage de trajectoires linéaires, à l'aide de la suite des côtés des polygones croisés. On donne une caractérisation complète de l'adhérence de l'ensemble des suites de coupage, dans l'esprit de la caractérisation classique des suites sturmiennes et de la récente caractérisation par Smillie-Ulcigrai des suites de coupage des trajectoires linéaires dans les polygones réguliers. La caractérisation est donnée en termes d'un système fini de substitutions (connu aussi sous le nom de présentation $\mathcal{S}$-adique), réglé par une transformation unidimensionnelle qui ressemble à l'algorithme de fraction continue. Comme dans le cas sturmien et dans celui des polygones réguliers, la caractérisation est basée sur la renormalisation et sur la définition d'un opérateur combinatoire de dérivation approprié. Une des nouveautés est que la dérivation se fait en deux étapes, sans utiliser directement les éléments du groupe de Veech, mais en utilisant un difféomorphisme affine qui envoie une surface de Bouw-Möller vers sa surface « duale », qui est dans le même disque de Teichmüller. Un outil technique utilisé est la présentation des surfaces de Bouw-Möller par les diagrammes de Hooper.

doi:10.24033/asens.2401

1. Introduction

In this paper we give a complete characterization of a class of symbolic sequences that generalizes the famous class of *Sturmian sequences*, that arises geometrically by coding bi-infinite linear trajectories on Bouw-Möller surfaces. A gentle introduction for the non-familiar reader is given below, but first we will give a short version of our main result. The *Bouw-Möller* family of translation surfaces is a family of Veech surfaces (see § 2 for definitions) indexed by two parameters, (m, n), so that the Bouw-Möller surface $\mathrm{S}_{m,n}$ is obtained by identifying parallel sides of m semi-regular polygons with symmetry of order n (the definition is given in § 2.5). Let $\mathscr{A}_{m,n}$ be an alphabet that labels (pairs of identified) sides of these polygons, and let $w \in \mathscr{A}_{m,n}^{\mathbb{Z}}$ be a sequence that codes a bi-infinite linear trajectory on $\mathrm{S}_{m,n}$ (called a *cutting sequence*). Our main result characterizes the closure of the set of such sequences in $\mathscr{A}_{m,n}^{\mathbb{Z}}$ in terms of a finite family of substitutions as follows (see § 9.3 for the definition of a substitution).

THEOREM 1.1. – *For any Bouw-Möller surface $\mathrm{S}_{m,n}$, there exist $(m-1)(n-1)$ substitutions $\sigma_i = \sigma_i^{m,n}$, $i = 1, \ldots, (m-1)(n-1)$ on an alphabet $\mathscr{A}'_{m,n}$, and operators $\mathbf{T}_i = \mathbf{T}_i^{m,n}$, $i = 0, \ldots, 2n-1$ from sequences in ${\mathscr{A}'_{m,n}}^{\mathbb{Z}}$ to sequences in $\mathscr{A}_{m,n}^{\mathbb{Z}}$, such that the following characterization holds:*

A sequence $w \in \mathscr{A}_{m,n}^{\mathbb{Z}}$ is in the closure of the set of cutting sequences of bi-infinite linear trajectories on the Bouw-Möller surface $\mathrm{S}_{m,n}$ if and only if there exists a sequence $(s_k)_{k\in\mathbb{N}}$ of indices $1 \le s_k \le (m-1)(n-1)$, $0 \le s_0 \le 2n-1$, and a sequence of letters $a_k \in \mathscr{A}'_{m,n}$, such that w can be written as [(1)]

$$w = \lim_{k\to\infty} \mathbf{T}_{s_0} \circ \sigma_{s_1} \circ \sigma_{s_2} \circ \cdots \circ \sigma_{s_k}(a_k). \tag{1}$$

The above expression is called an $\mathcal{S}$*−adic expansion* of the word w, and this type of characterization, which is well known in the world of word combinatorics, is known as an $\mathcal{S}$*−adic characterization* (see for example [7] or [1]). We emphasize that the notion of $\mathcal{S}$−adic expansions is used to describe words with *low complexity* since, under some general assumptions, a word with an $\mathcal{S}$−adic presentation has *zero entropy* (see Theorem 4.3 in [7] for a precise statement). The substitutions $\sigma_i^{m,n}$ and the operators $\mathbf{T}_i^{m,n}$ are explicitly constructed in the paper [(2)].

Furthermore, in this paper we show that the sequence $(s_k)_{k\in\mathbb{N}}$ that appears in the $\mathcal{S}$−adic expansion of w is the itinerary of a certain Farey-type map $\mathcal{F}_{m,n}$ on the set of directions. In particular, it is completely determined by knowing the direction of the trajectory coded by the cutting sequence w. This can be used in two ways: on one hand, given a direction $\theta \in S^1$, one can hence algorithmically produce, by iterating our substitutions, all (finite length blocks of) cutting sequences of trajectories in direction θ. On the other hand, given a sequence w that is the cutting sequence of a trajectory in an unknown direction θ, one can recover the

(1) The limit is taken along a sequence of *finite* words, for which convergence in $\mathscr{A}_{m,n}^{\mathbb{Z}}$ to the infinite word w means that the finite words, as k grows, share larger and larger central blocks of letters.

(2) A technical detail is that this $\mathcal{S}$−adic presentation is possible only at the level of *transitions*, namely pairs of consecutive letters in w; indeed the alphabet $\mathscr{A}'_{m,n}$ on which the substitutions are defined is an alphabet labeling transitions, and the operators $\mathbf{T}_i^{m,n}$ (defined in § 9.3) simply transform a sequence of transitions into a sequence of letters in $\mathscr{A}_{m,n}$.

sequence $(s_k)_{k\in\mathbb{N}}$ from w (see the informal discussion after Theorem 1.5 in this introduction and Section § 7.5 for more details) and hence use the map $\mathcal{F}_{m,n}$ to recover (uniquely, if w is non-periodic) the direction θ (see Proposition 8.6).

In order to introduce the problem of characterization of cutting sequences and motivate the reader, we start this introduction by recalling in § 1.1 the geometric construction of *Sturmian sequences* in terms of coding linear trajectories in a square, and then both their characterization using derivation, as described by Series, and their $\mathcal{S}$−adic presentation by a system of substitutions. We then recall in § 1.2 how this type of description was recently generalized by several authors to the sequences coding linear trajectories in regular polygons. Finally, in § 1.3 we explain why Bouw-Möller sequences are the next natural example to consider to extend these symbolic characterizations, and state a simple case of our main result.

1.1. Sturmian sequences

Sturmian sequences are an important class of sequences in two symbols that often appear in mathematics, computer science and real life. They were considered by Christoffel [10] and Smith [39] in the 1870's, by Morse and Hedlund [31] in 1940 and by many authors since then (see [1] for a contemporary account and [26] for a historical survey). Sturmian sequences are interesting because of their geometric origin, and are also of interest because they give the simplest non-periodic infinite sequences (see [11]), having the lowest possible complexity. (3) They admit the following geometric interpretation:

Consider an *irrational line*, i.e., a line in the plane in a direction θ such that $\tan\theta$ is irrational, in a *square grid* (Figure 1). As we move along the line, let us record with a 0 each time we hit a horizontal side and with a 1 each time we hit a vertical side. We get in this way a bi-infinite sequence of 0s and 1s which, up to choosing an origin arbitrarily, we can think of as an element in $\{0, 1\}^{\mathbb{Z}}$. The sequences obtained in this way as the line vary among all possible irrational lines are exactly all *Sturmian sequences*. (For further reading, see the beautiful expository paper by Series [34], and also the introduction of [38].)

Equivalently, by looking at a fundamental domain of the periodic grid, we can consider a square with opposite sides identified by translations. We define a *linear trajectory* in direction θ to be a path that starts in the interior of the square and moves with constant velocity vector making an angle θ with the horizontal, until it hits the boundary, at which time it re-enters the square at the corresponding point on the opposite side and continues traveling with the same velocity. For an example of a trajectory see Figure 1. We will restrict ourselves to trajectories that do not hit vertices of the square. As in Figure 1, let us label by 0 and 1 respectively its horizontal and vertical sides. (4) The *cutting sequence* $c(\tau)$ associated to the linear trajectory τ is the bi-infinite word in the symbols (edge labels, here 0 and 1) of the alphabet $\mathscr{L}$, which is obtained by reading off the labels of the pairs of identified sides crossed by the trajectory τ as time increases.

(3) For each n let $P(n)$ be the number of possible strings of length n. For Sturmian sequences, $P(n) = n + 1$.
(4) Since squares (or, more generally, parallelograms) tile the plane by translation, the cutting sequence of a trajectory in a square (parallelogram) is the same than the cutting sequence of a straight line in $\mathbb{R}^2$ with respect to a square (or affine) grid.