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COHERENT COHOMOLOGY OF SHIMURA VARIETIES AND AUTOMORPHIC FORMS

BY JUN SU

ABSTRACT. – We show that the cohomology of canonical extensions of automorphic vector bundles over toroidal compactifications of Shimura varieties can be computed by relative Lie algebra cohomology of automorphic forms. Our result is inspired by and parallel to Borel and Franke's work on the cohomology of automorphic local systems on locally symmetric spaces, and also generalizes a theorem of Mumford.

RÉSUMÉ. – Nous montrons que la cohomologie des extensions canoniques des fibrés vectoriels automorphes sur les compactifications toroïdales des variétés de Shimura peut être calculée par la cohomologie relative des algèbres de Lie des formes automorphes. Notre résultat est inspiré par les travaux de Borel et Franke sur la cohomologie des systèmes locaux automorphes sur les espaces localement symétriques, et généralise également un théorème de Mumford.

Introduction

The cohomology of Shimura varieties are of interest and importance in the Langlands program as their Galois representations and Hecke modules meet each other (for the first time). Let (G, X) be a Shimura datum, $\mathrm{Sh}(G, X)_{\mathbb{K}}$ be the associated Shimura variety at a neat level $\mathbb{K}$ and $\mathrm{Sh}_{\mathbb{K}}$ be its complex analytification, then there are the Betti cohomology $H^*(\mathrm{Sh}_{\mathbb{K}}, \mathbb{C})$ and more generally cohomology of local systems on $\mathrm{Sh}_{\mathbb{K}}$ arising from algebraic representations of G. In this case the Galois representations spring from étale cohomology, while on the other hand at first it is in general unknown whether the natural Hecke module structures on these cohomology come from automorphic representations of G. This question can actually be asked for every locally symmetric space: for simplicity we assume the locally symmetric space is of the form

$$S_{\mathbb{K}} := G(\mathbb{Q})\backslash G(\mathbb{A})/K\mathbb{K},$$

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where G is a semisimple group over $\mathbb{Q}$, K is an open subgroup of a maximal compact subgroup of $G(\mathbb{R})$ and $\mathbb{K} \subseteq G(\mathbb{A}_f)$ is a neat compact open subgroup, then every finite-dimensional representation E of $G(\mathbb{R})$ defines a local system

$$\underline{E} := G(\mathbb{Q})\backslash((G(\mathbb{A})/K\mathbb{K}) \times E)$$

on $S_{\mathbb{K}}$ (which we will refer to as an *automorphic local system*). There are standard Hecke-equivariant isomorphisms

$$H^i(S_{\mathbb{K}}, \underline{E}) \cong H^i_{(\mathfrak{g},K)}(C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{K})^{K\text{-fin}} \otimes E), \tag{0.1}$$

where $\mathfrak{g}$ is the complexified Lie algebra of G. Borel [8], [7] conjectured that in (0.1) $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{K})$ can be replaced by $\mathcal{A}(G)^{\mathbb{K}}$, i.e., we have

$$H^i(S_{\mathbb{K}}, \underline{E}) \cong H^i_{(\mathfrak{g},K)}(\mathcal{A}(G)^{\mathbb{K}} \otimes E), \tag{0.2}$$

where $\mathcal{A}(G)$ denotes the space of automorphic forms on G. Note that in the above settings there is nothing special about $\mathbb{Q}$ among all number fields: for G over any number field F, the locally symmetric spaces and the automorphic sheaves thereon arising from G or $\mathrm{Res}_{F/\mathbb{Q}}G$ are the same, while $\mathcal{A}(G)$ and $\mathcal{A}(\mathrm{Res}_{F/\mathbb{Q}}G)$ are canonically isomorphic as Hecke modules. The difficulty of the conjecture is rooted in the non-compactness of $S_{\mathbb{K}}$ and hence the cases where G is anisotropic over $\mathbb{Q}$ are somewhat straightforward. Borel himself showed that in (0.1) one can replace $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{K})^{K\text{-fin}}$ with the space $C^\infty_{\mathrm{umg}}(G)^{\mathbb{K}}$ of K-finite smooth functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{K}$ with uniformly moderate growth (see Definition 2.4) [7, 3.2], [9, Theorem 1], where the latter differs from $\mathcal{A}(G)^{\mathbb{K}}$ by a finiteness condition under differentiation by elements of the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$. Harder and Casselman-Speh solved the rank-1 cases of the conjecture, and the full conjecture was eventually proved by Franke in his famous paper [21].

For Shimura varieties it is also natural to consider coherent sheaf cohomology groups like $H^q(\mathrm{Sh}_{\mathbb{K}}, \Omega^p)$. A suitable class of coefficients are the automophic vector bundles introduced by Harris and Milne in [23] and [36]. These vector bundles are parametrized by representations of a parabolic subgroup P_h (arising from a $h \in X$) of $G_{\mathbb{C}}$ and they are analytifications of algebraic vector bundles defined over number fields specified by the representations. They have the name as every holomorphic or nearly holomorphic (vector-valued) automorphic form can be intepreted as a global section of one of them. The family of these vector bundles is closed under tensor operations and includes the sheaves $\Omega^p_{\mathrm{Sh}_{\mathbb{K}}}$ of holomorphic $(p,0)$-forms for all p. However, the cohomology of these vector bundles may not always be useful. The basic example is that in the modular curve case these cohomology groups are torsion-free modules over the ring of polynomials in the j-invariant and hence are either 0 or infinite-dimensional. In general for an automorphic vector bundle $\widetilde{V}$ arising from a representation V of P_h, analogous to (0.1) we have

$$H^i(\mathrm{Sh}_{\mathbb{K}}, \widetilde{V}) \cong H^i_{(\mathfrak{p}_h,K_h)}(C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{K}) \otimes V)$$

(see (1.11)), where $\mathfrak{p}_h$ is the Lie algebra of P_h, $K_h \subseteq G(\mathbb{R})$ is the stabilizer of a point in X and when K_h is non-compact $H^i_{(\mathfrak{p}_h,K_h)}(-)$ abusively denotes cohomology of the complex [30, 2.127] usually used to compute relative Lie algebra cohomology. In contrast to (0.2) the right hand side above could be strictly larger than

$$H^i_{(\mathfrak{p}_h,K_h)}(\mathcal{A}(G)^{\mathbb{K}} \otimes V).$$

In fact by [32, 9.2,10.1] and the main Theorem 5.7 of this paper, when i is 1 less than the codimension of the boundary in the Baily-Borel compactification this happens for every $\widetilde{V}$ after being twisted by a sufficiently high power of the canonical bundle. A remedy for this problem is built upon the theory of toroidal compactifications of locally symmetric varieties introduced by Ash, Mumford, Rapoport and Tai in [2] (following the work of Igusa on Siegel modular varieties and Hirzebruch on Hilbert modular surfaces) and the canonical extensions of automorphic vector bundles over these compactifications introduced by Mumford and Harris in [37] and [24]. The aim of this paper is to show that after applying these constructions we get favorable cohomology groups (see Theorem 5.7):

THEOREM. – *Let $\widetilde{V}$ be an automorphic vector bundle over* $\mathrm{Sh}_{\mathbb{K}}$ *arising from a representation V of P_h, $\widetilde{V}^{\mathrm{can}}$ be its canonical extension over an admissible toroidal compactification* $\mathrm{Sh}_{\mathbb{K},\Sigma}$ *of* $\mathrm{Sh}_{\mathbb{K}}$*, then there are Hecke-equivariant isomorphisms*

$$H^i(\mathrm{Sh}_{\mathbb{K},\Sigma}, \widetilde{V}^{\mathrm{can}}) \cong H^i_{(\mathfrak{p}_h, K_h)}(\mathcal{A}(G)^{\mathbb{K}} \otimes V). \tag{0.3}$$

We note that the insight to consider the left hand side above originated in Harris' paper [26]. Toroidal compactifications of a locally symmetric variety are usually not unique and form an inverse system indexed by some combinatorial data Σ, but the cohomology of canonical extensions of an automorphic vector bundle to different toroidal compactifications are naturally isomorphic. There are plenty of toroidal compactifications $\mathrm{Sh}_{\mathbb{K},\Sigma}$ of $\mathrm{Sh}_{\mathbb{K}}$ that are smooth and such that the boundary $Z := \mathrm{Sh}_{\mathbb{K},\Sigma}\backslash\mathrm{Sh}_{\mathbb{K}}$ is a normal crossings divisor. In this case $(\Omega^p_{\mathrm{Sh}_{\mathbb{K}}})^{\mathrm{can}}$ is the sheaf $\Omega^p_{\mathrm{Sh}_{\mathbb{K},\Sigma}}(\log Z)$ of holomorphic $(p,0)$-forms on $\mathrm{Sh}_{\mathbb{K}}$ with log poles along Z, whose cohomology groups appear a lot in algebraic geometry. For instance, global sections of the log canonical bundle $\omega_{\mathrm{Sh}_{\mathbb{K},\Sigma}}(Z)$ and its powers make the log canonical ring of Iitaka of $\mathrm{Sh}_{\mathbb{K}}$. This led Mumford to compute the global sections of canonical extensions of general automorphic vector bundles and obtain the degree 0 case of our theorem as [37, Proposition 3.3]. As another example, the groups $H^q(\mathrm{Sh}_{\mathbb{K},\Sigma}, \Omega^p(\log Z))$ form the E_1 page of a spectral sequence computing $H^*(\mathrm{Sh}_{\mathbb{K}}, \mathbb{C})$ and are thus related to the mixed Hodge structures attached to $\mathrm{Sh}_{\mathbb{K}}$. In fact, Faltings [17], [18] had independently suggested studying the higher cohomology of canonical extensions and has since demonstrated their importance for the study of the Hodge structures attached to Siegel modular forms [1].

From the representation theoretic point of view, an advantage of coherent cohomology over the cohomology of automorphic local systems is that the right hand side of (0.3) detects some more automorphic representations than the right hand side of (0.2). The examples include weight 1 modular forms and every cuspidal automorphic representation whose archimedean factor is a non-degenerate limit of discrete series. Our result in particular implies the algebraicity of Hecke eigenvalues arising from these automorphic representations. The attachment of Galois representations to those automorphic representations or Hecke eigenclasses in coherent cohomology is an interesting and important project. In degree 0, Deligne-Serre [14] did it for weight 1 modular forms, and Taylor [41] took care of Siegel modular forms of low weights. The interest in associating Galois representations to eigenclasses in higher coherent cohomology was recently refueled by the work [11] of Calegari-Geraghty, where the full power of their result partly relies on the existence of