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*Free $\mathbb{Q}$-groups are residually torsion-free nilpotent*

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# FREE $\mathbb{Q}$-GROUPS ARE RESIDUALLY TORSION-FREE NILPOTENT

BY ANDREI JAIKIN-ZAPIRAIN

ABSTRACT. – We develop a method to show that some (abstract) groups can be embedded into a free pro-$p$ group. In particular, we show that every finitely generated subgroup of a free $\mathbb{Q}$-group can be embedded into a free pro-$p$ group for almost all primes $p$. This solves an old problem raised by G. Baumslag: free $\mathbb{Q}$-groups are residually torsion-free nilpotent.

RÉSUMÉ. – Nous développons une méthode pour montrer que certains groupes (discrets) peuvent être plongés dans un pro-$p$-groupe libre. Nous montrons en particulier que tout sous-groupe de type fini d'un $\mathbb{Q}$-groupe libre peut être plongé dans un pro-$p$-groupe libre pour presque tous les premiers $p$. Cela résout un ancien problème soulevé par G. Baumslag: les $\mathbb{Q}$-groupes libres sont résiduellement nilpotents sans torsion.

## 1. Introduction

A group $G$ is called a $\mathbb{Q}$-*group* if for any $n \in \mathbb{N}$ and $g \in G$ there exists exactly one $h \in G$ satisfying $h^n = g$. These groups were introduced by G. Baumslag in [4] under the name of $\mathcal{D}$-groups. He observed that $\mathbb{Q}$-groups may be viewed as universal algebras, and as such they constitute a variety. Every variety of algebras contains free algebras (in that variety). In the variety of $\mathbb{Q}$-groups we call such free algebras *free* $\mathbb{Q}$-*groups*. G. Baumslag dedicated several papers to the study of residual properties of free $\mathbb{Q}$-groups [5, 7, 9]. For example, in [5] he showed that a free $\mathbb{Q}$-group is residually periodic-by-soluble and locally residually finite-by-soluble. He wrote in [5] "It is, of course, still possible that, locally, free $\mathcal{D}$-groups are, say, residually finite $p$-groups" or in [7] "In particular it seems likely that free $\mathcal{D}$-groups are residually torsion-free nilpotent. However the complicated nature of free $\mathcal{D}$-groups makes it difficult to substantiate such a remark." This conjecture is part of two main collections of problems in group theory ([10, Problem F12] and [38, Problem 13.39 (a),(c)]), and in addition to mentioned works of Baumslag, it was also studied in [16, 22]. In this paper we solve Baumslag's conjecture.


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THEOREM 1.1. – *A free $\mathbb{Q}$-group is residually torsion-free nilpotent.*

The structure of a finitely generated subgroup of a free $\mathbb{Q}$-group was studied already in [4] (see also [53, Section 8] and Proposition 5.3). It was shown that it is the end result of repeatedly freely adjoining $n$th roots to a finitely generated free group. The key point of our proof of Theorem 1.1 is to show that any finitely generated subgroup of a free $\mathbb{Q}$-group can be embedded into a finitely generated free pro-$p$ group for some prime $p$. We actually prove the following more precise result.

THEOREM 1.2. – *Let $p$ be a prime. Let $H_0$ be a finitely generated free group and let $H_0 \hookrightarrow \mathbf{F}$ be the canonical embedding of $H_0$ into its pro-$p$ completion $\mathbf{F}$. Let $(H_i)_{i\geq 0}$ be a sequence of subgroups of $\mathbf{F}$ such that for $i \geq 0$,*

1. *$H_{i+1} = \langle H_i, B_i\rangle$, where $B_i$ is a finitely generated abelian subgroup of $\mathbf{F}$ and*
2. *$A_i = H_i \cap B_i$ is a maximal abelian subgroup of $H_i$.*

*Then for every $i \geq 0$, the canonical map*

$$H_i *_{A_i} B_i \to H_{i+1}$$

*is an isomorphism,*

Theorem 1.2 is actually an application of the slightly more technical Theorem 5.1.

Let us make a few remarks about the groups $A_i$ and $B_i$. It is relatively easy to describe abelian subgroups of amalgamated products. In particular, the conclusion of the theorem implies that all abelian subgroups of $H_i$ are finitely generated. Thus, an implicit hypothesis, which appears in the theorem, that maximal abelian subgroups $A_i$ of $H_i$ are finitely generated, is automatically fulfilled.

A maximal abelian subgroup of $\mathbf{F}$ is isomorphic to the additive group of the ring of $p$-adic numbers $(\mathbb{Z}_p, +)$. Therefore, for any finitely generated (abstract) abelian subgroup $A$ of $\mathbf{F}$ and any finitely generated torsion-free abelian group $B$ which contains $A$ and such that $B/A$ has no $p$-torsion, it is posible to extend the embedding $A \hookrightarrow \mathbf{F}$ to an embedding $B \hookrightarrow \mathbf{F}$. This extension is unique if and only if $B/A$ is finite.

Given a commutative ring $A$, we will introduce in Section 5 the notions of $A$-group and free $A$-group $F^A(X)$. For example, a free pro-$p$ group is an example of a $\mathbb{Z}_p$-group. We have the following consequence of Theorem 1.2.

COROLLARY 1.3. – *Let $F(X)$ be the free group on a finite free generating set $X$, let $\mathbf{F}$ be its pro-$p$ completion. Then the canonical homomorphism*

$$\phi : F^{\mathbb{Z}_p}(X) \to \mathbf{F}$$

*is injective.*

Let $H$ be a group and $A$ the centralizer of a non-trivial element. Then the group $G = H *_A (A \times \mathbb{Z}^k)$ is said to be obtained from $H$ by *extension of a centralizer*. A group is called an *ICE* group if it can be obtained from a free group using iterated centralizer extensions. A group $G$ is a *limit group* if and only if it is a finitely generated subgroup of an ICE group (see [37, 15]). All centralizers of non-trivial elements of an ICE group are abelian. Thus, Theorem 1.1 provides explicit realizations of ICE groups (and so limit groups) as subgroups of a non-abelian free pro-$p$-group (for this application we only need the case where all $B_i/A_i$ are torsion-free). Non-explicit realizations of limit groups as subgroups of a non-abelian free pro-$p$ group (in fact, as subgroups of every compact group containing a non-abelian free group) were obtained in [2] (see also [13]).

In Section 5 we recall the definition of the $\mathbb{Q}$-completion of a group $G$. For example, a free $\mathbb{Q}$-group is the $\mathbb{Q}$-completion of a free group. Theorem 1.2 allows also to show that the $\mathbb{Q}$-completion of a limit group is residually torsion-free nilpotent.

THEOREM 1.4. – *The $\mathbb{Q}$-completion of a limit group is residually torsion-free nilpotent.*

A group $G$ is called *parafree* if it is residually nilpotent and for some free group $F$, we have that for all $i$, $G/\gamma_i(G) \cong F/\gamma_i(F)$ where $\gamma_i(G)$ denotes the terms of the lower central series of $G$. Baumslag introduced this family of groups and produced many examples of them [6]. In [35] we apply the method of the proof of Theorem 1.2 in order to construct new examples of finitely generated parafree groups.

Our proof of Theorem 1.2 is by induction on $i$. In the inductive step argument we start with the following situation. We have a finitely generated subgroup $H$ of $\mathbf{F}$, a maximal abelian subgroup $A$ of $H$ and an abelian subgroup $B$ of $\mathbf{F}$ containing $A$. We want to show that the canonical homomorphism $H *_A B \to \langle H, B\rangle$ is an isomorphism. Unfortunately, we do not know how to show this statement in such a generality, but we prove it in Theorem 5.1 under an additional assumption that the embedding $H \hookrightarrow \mathbf{F}$ is strong (see Definition 3.8). Theorem 5.1 is the main result of the paper. Its proof uses in an essential way the results of [32], where we proved a particular case of the Lück approximation in positive characteristic.

The paper is organized as follows. In Section 2 we give basic preliminaries. The proof of Theorem 5.1 uses the theory of mod-$p$ $L^2$-Betti numbers. In Section 3 we explain how to define them for subgroups $G$ of a free pro-$p$ group. In Section 4 we introduce a technical notion of $\mathcal{D}$-torsion-free modules and show that some relevant $\mathbb{F}_p[G]$-modules are $\mathcal{D}_{\mathbb{F}_p[G]}$-torsion-free (see Proposition 4.10). In Section 5 we prove Theorem 5.1 and obtain all the results mentioned in the introduction. In Section 6 we discuss the following two well-known problems concerning linearity of free pro-$p$ groups and free $\mathbb{Q}$-groups:

QUESTION 1.5. – 1. *(I. Kapovich) Is a free $\mathbb{Q}$-group linear?*

2. *(A. Lubotzky) Is a free pro-p group linear?*

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