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Dustin CLAUSEN, Akhil MATHEW, Niko NAUMANN
& Justin NOEL

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : `annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
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Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : `abonnements@smf.emath.fr`

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DESCENT AND VANISHING IN CHROMATIC ALGEBRAIC K-THEORY VIA GROUP ACTIONS

BY Dustin CLAUSEN, Akhil MATHEW, Niko NAUMANN AND Justin NOEL

ABSTRACT. – We prove some K-theoretic descent results for finite group actions on stable ∞-categories, including the p-group case of the Galois descent conjecture of Ausoni-Rognes. We also prove vanishing results in accordance with Ausoni-Rognes's redshift philosophy: in particular, we show that if R is an $\mathbb{E}_\infty$-ring spectrum with $L_{T(n)}R = 0$, then $L_{T(n+1)}K(R) = 0$. Our key observation is that descent and vanishing are logically interrelated, permitting to establish them simultaneously by induction on the height.

RÉSUMÉ. – Nous démontrons quelques résultats sur la descente K-théorique pour des actions de groupes finis sur des ∞-catégories stables, dont le cas des p-groupes de la conjecture de descente galoisienne d'Ausoni-Rognes. Nous obtenons aussi des résultats d'annulation en accord avec la philosophie de « décalage vers le rouge » d'Ausoni-Rognes: en particulier, nous démontrons que si R est un $\mathbb{E}_\infty$-anneau avec $L_{T(n)}R = 0$, alors on a $L_{T(n+1)}K(R) = 0$. Notre observation-clef est que la descente et l'annulation sont logiquement liées, ce qui permet de les établir simultanément par récurrence sur la hauteur.

1. Introduction

In this paper, we prove some results concerning the algebraic K-theory of ring spectra and stable ∞-categories after $T(n)$-localization. Throughout this paper, our telescopes $T(n)$ are taken at a fixed implicit prime p and height $n \geq 0$; we adopt the convention $T(0) = \mathbb{S}[1/p]$. Our starting point is the following two results concerning classical commutative rings R:

THEOREM 1.1 ([59]). – *For $n \geq 2$, we have $L_{T(n)}K(R) = 0$.*

THEOREM 1.2 ([70], [71], [23]). – *For G a finite group and $R \to R'$ a G-Galois extension, the natural comparison map $L_{T(1)}K(R) \to (L_{T(1)}K(R'))^{hG}$ is an equivalence.*

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Thus, the K-theory of an ordinary commutative ring has no chromatic information beyond height one, and the localization to height one is well-behaved in its descent properties. In fact, $T(1)$-local K-theory is even better-behaved than suggested by Theorem 1.2: under mild finiteness hypotheses, the Galois descent can be upgraded to an étale hyperdescent result, which leads to a descent spectral sequence from étale cohomology to $T(1)$-local K-theory as produced by [70, 71]. Furthermore, one knows that under such conditions, the map $K(R;\mathbb{Z}_p) \to L_{T(1)}K(R)$ from p-adic K-theory to its $T(1)$-localization is an equivalence in high enough degrees, i.e., one has the Lichtenbaum-Quillen conjecture, thanks to the work of Voevodsky-Rost, cf. [65, 22] for accounts. However, we will not touch on these more advanced aspects in this paper.

Moving from ordinary rings to more general ring spectra, Ausoni-Rognes suggested that the above two theorems should fit into a broader "redshift" philosophy in algebraic K-theory, [7, 8]. For an $\mathbb{E}_1$-ring spectrum R, one expects that taking algebraic K-theory increases the "chromatic complexity" of R by one. In the setting of Theorem 1.1, the Eilenberg-MacLane spectrum HR has no chromatic information at heights ≥ 1, while the result states that $K(R) = K(HR)$ has no chromatic information at heights ≥ 2; furthermore, Theorem 1.2 and its refinement to hyperdescent control the height one information very precisely.

For $\mathbb{E}_\infty$-rings R, there is a particularly well-behaved notion of chromatic complexity, thanks to a theorem of Hahn [32]: if $L_{T(n)}R = 0$, then $L_{T(m)}R = 0$ also for all $m > n$. If R is an $\mathbb{E}_\infty$-ring, then so is $K(R)$, and in this setting one possible expression of the redshift philosophy would be that $L_{T(n)}R = 0 \Leftrightarrow L_{T(n+1)}K(R) = 0$. Here we prove half of this statement.

Theorem A. – *Let R be an $\mathbb{E}_\infty$-ring and $n \geq 0$. If $L_{T(n)}R = 0$, then $L_{T(n+1)}K(R) = 0$.*

Recent work of Burklund-Schlank-Yuan [20, Th. 9.11] and Yuan [73] proves the converse of Theorem A: if R is a p-local $\mathbb{E}_\infty$-ring with $L_{T(n)}R \neq 0$, then $L_{T(n+1)}K(R) \neq 0$. Many special cases of Theorem A were previously known. In particular, in important specific cases, much more precise (Lichtenbaum-Quillen) statements about $K(R)$ have been proved, as in [34, 2, 33, 6, 7].

Theorem A generalizes Mitchell's vanishing Theorem 1.1. We note that there is a more general statement which applies also to $\mathbb{E}_1$-rings A: if both $L_{T(n)}A = 0$ and $L_{T(n+1)}A = 0$, then $L_{T(n+1)}K(A) = 0$; see Corollary 4.11, which is also explored in [43].

We also have an analog of Thomason's descent Theorem 1.2. For the statement, we need to assume $T(n)$-local vanishing of the C_p-Tate construction R^{tC_p} (taken with respect to the trivial action); this assumption is satisfied if R is a discrete ring and $n = 1$, i.e., the setting of Theorem 1.2. In addition, we need to assume the finite group G is a p-group, where p is the (throughout fixed) prime at which chromatic localizations are taken.

Theorem B. – *Let R be an $\mathbb{E}_\infty$-ring and $n \geq 0$. Suppose $L_{T(n)}(R^{tC_p}) = 0$ for the fixed prime p. Then for $\mathcal{C}$ any R-linear idempotent-complete stable ∞-category equipped with an R-linear action of a finite p-group G, the homotopy fixed point comparison map for $T(n+1)$-local K-theory is an equivalence:*

$$L_{T(n+1)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} (L_{T(n+1)}K(\mathcal{C}))^{hG}. \tag{1.1}$$

If $R \to R'$ is a G-Galois extension of commutative rings, then by Galois descent we have $\mathrm{Perf}(R) \xrightarrow{\sim} \mathrm{Perf}(R')^{hG}$; thus, when $n = 0$, Theorem B recovers the p-group case of Theorem 1.2. But in fact the case of general G in Theorem 1.2 reduces to the p-group case by a simple transfer argument, as already pointed out and exploited by Thomason; in particular, Theorem B implies Theorem 1.2.

However, Theorem B does not hold for an arbitrary finite group G, essentially because the G-action is allowed to be arbitrary. In fact, for the trivial action of G on $\mathrm{Perf}(\mathbb{C})$ and $n = 0$, one can calculate both sides of (1.1) using Suslin's equivalence [67] between topological and algebraic K-theory. One obtains that the source is the p-completed G-equivariant topological K-theory of a point while the target is $\mathrm{KU}_{\hat{p}}^{BG}$. For G of prime-to-p order the result is evidently false because $\mathrm{KU}_{\hat{p}}^{BG} = \mathrm{KU}_{\hat{p}}$, while for G a p-group, Theorem B amounts to the p-complete Atiyah-Segal completion theorem. Nonetheless, there is a generalization of Theorem B to arbitrary finite groups which shows that the descent question for arbitrary G reduces to that for cyclic subgroups of prime-to-p order; see Theorem 6.5.

We remark that these theorems also hold with K-theory replaced by an arbitrary additive invariant, and one also has "co-descent" or "assembly map" equivalences dual to the descent statements of Theorem B; see Proposition 4.1 for more details.

Let us now give the basic example of these results. Throughout this paper, we will use the notation $L_n^{p,f} = L_{T(0)\oplus\cdots\oplus T(n)}$, following [43]; in particular, we have the $L_n^{p,f}$-local sphere $L_n^{p,f}\mathbb{S}$. An $L_n^{p,f}$-local stable ∞-category is one where the mapping spectra are $L_n^{p,f}$-local, or equivalently one which is $L_n^{p,f}\mathbb{S}$-linear. By Kuhn's "blueshift" theorem [42], if a spectrum X is $L_n^{p,f}$-local then X^{tC_p} is $L_{n-1}^{p,f}$-local. Thus, from Theorem A and Theorem B we deduce the following:

THEOREM C. – *Let $n \geq 0$, and let $\mathcal{C}$ be an $L_n^{p,f}$-local idempotent-complete stable ∞-category. Then $L_{T(m)}K(\mathcal{C}) = 0$ for all $m \geq n+2$, and for any finite p-group G acting on $\mathcal{C}$ we have*

$$L_{T(n+1)}K(\mathcal{C}^{hG}) \xrightarrow{\sim} (L_{T(n+1)}K(\mathcal{C}))^{hG}.$$

In fact, for the proofs of Theorem A and Theorem B we proceed by first proving this special case, Theorem C. Then we combine with a recent result of Land-Mathew-Meier-Tamme [43] to the effect that $L_{T(n)}K(R) \xrightarrow{\sim} L_{T(n)}K(L_n^{p,f}R)$ (for $n \geq 1$) which lets us deduce the general case. (Actually, we also use the result of [43] in the proof of Theorem C, but in a more indirect way.)

It turns out that our arguments establish a logical connection between the vanishing and the descent theorems. This is expressed in the following result, from which we deduce all of the above theorems.

THEOREM 1.3 (Inductive vanishing, Lemma 4.9). – *Let R be an $\mathbb{E}_\infty$-ring spectrum and $n \geq 1$. Then for the following conditions, we have the implications* (A) $\Rightarrow$ (B) $\Rightarrow$ (C)*:*

(A) $L_{T(n)}R = 0$ *and* $L_{T(n)}K(R^{tC_p}) = 0$.