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ABSOLUTELY CONTINUOUS SELF-SIMILAR MEASURES WITH EXPONENTIAL SEPARATION

BY Samuel KITTLE

ABSTRACT. – In this paper, we present a sufficient condition for a self-similar measure to be absolutely continuous. In the special case of Bernoulli convolutions, we show that the Bernoulli convolution with algebraic parameter λ is absolutely continuous provided λ satisfies a simple condition in terms of the Mahler measure of λ, its Garsia entropy and λ. Using this, we are able to give examples of λ for which the Bernoulli convolution with parameter λ is absolutely continuous and for which λ is not close to 1.

RÉSUMÉ. – Dans cet article, nous présentons une condition suffisante pour qu'une mesure auto-similaire soit absolument continue. Dans le cas particulier des convolutions de Bernoulli, nous montrons que la convolution de Bernoulli de paramètre algébrique lambda est absolument continue à condition que lambda satisfasse une condition simple en termes de mesure de Mahler de lambda, son entropie de Garsia et lambda. Grâce à cela, nous pouvons donner des exemples de lambda pour lesquels la convolution de Bernoulli de paramètre lambda est absolument continue et pour lesquels lambda n'est pas proche de 1.

1. Introduction

1.1. Statement of results for Bernoulli convolutions

The main result of this paper is to give a sufficient condition for a self-similar measure to be absolutely continuous. For simplicity, we first state this result in the case of Bernoulli convolutions. First we need to define Bernoulli convolutions.

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DEFINITION 1.1 (Bernoulli convolution). – Given some $\lambda \in (0,1)$, we define the Bernoulli convolution with parameter λ to be the law of the random variable Y given by

$$Y = \sum_{n=0}^{\infty} X_n \lambda^n,$$

where each of the X_n are i.i.d. random variables that have probability $\frac{1}{2}$ of being 1 and probability $\frac{1}{2}$ of being -1. We denote this measure by μ_λ.

Bernoulli convolutions are the most well studied examples of self-similar measures which are important objects in fractal geometry. We discuss these further in Section 1.2. Despite much effort, it is still not known for which λ the measure μ_λ is absolutely continuous. The results of this paper contribute towards answering this question.

DEFINITION 1.2 (Mahler measure). – Given some algebraic number α_1 with conjugates $\alpha_2, \alpha_3, \dots, \alpha_n$ whose minimal polynomial (over $\mathbb{Z}$) has leading coefficient C, we define the Mahler measure of α_1 to be

$$M_{\alpha_1} = |C| \prod_{i=1}^{n} \max\{|\alpha_i|, 1\}.$$

THEOREM 1.3. – *Let $\lambda \in \left(\frac{1}{2}, 1\right)$ be an algebraic number with Mahler measure M_λ. Suppose that λ is not the root of any non-zero polynomial with coefficients $0, \pm 1$ and satisfies*

$$(\log M_\lambda - \log 2)(\log M_\lambda)^2 < \frac{1}{27}(\log M_\lambda - \log \lambda^{-1})^3 \lambda^2. \tag{1}$$

Then the Bernoulli convolution with parameter λ is absolutely continuous.

This is a corollary of a more general statement about a more general class of self-similar measures which we discuss in Section 1.3. The requirement (1) is equivalent to $M_\lambda < F(\lambda)$ where $F : (\frac{1}{2}, 1) \to \mathbb{R}$ is some strictly increasing continuous function satisfying $F(\lambda) > 2$ and

$$(\log F(\lambda) - \log 2)\,(\log F(\lambda))^2 = \frac{1}{27}\left(\log F(\lambda) - \log \lambda^{-1}\right)^3 \lambda^2$$

for all $\lambda \in (\frac{1}{2}, 1)$. Figure 1 displays the graph of F.

It is worth noting that $F(\lambda) \to 2^{\frac{27}{26}} \approx 2.054$ as $\lambda \to 1$. The fact that $F(\lambda) > 2$ is important because the requirement that λ is not the root of a polynomial with coefficients $0, \pm 1$ forces $M_\lambda \geq 2$ as is explained in Remark 5.10.

Some parameters for Bernoulli convolutions which can be shown to be absolutely continuous using Theorem 1.3 are given in Table 1 which can be found in Section 6. The smallest value of λ that we were able to find for which the Bernoulli convolution with parameter λ can be shown to be absolutely continuous using this method is $\lambda \approx 0.78207$ with minimal polynomial $X^8 - 2X^7 - X + 1$. This is much smaller than the examples given in [20], the smallest of which was $\lambda = 1 - 10^{-50}$. We also show that for all $n \geq 8$, there is a root of the polynomial $X^n - 2X^{n-1} - X + 1$ which is in $\left(\frac{1}{2}, 1\right)$ such that the Bernoulli convolution with this parameter is absolutely continuous.

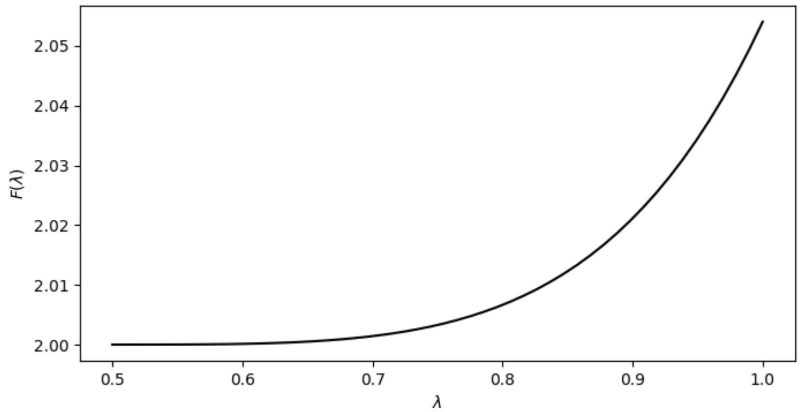

FIGURE 1. The graph of F

1.2. Review of existing literature

For a thorough survey on Bernoulli convolutions see Peres–Schlag–Solomyak [14] or Solomyak [18]. For a review of recent developments see Varjú [19] or Hochman [6]. Bernoulli convolutions were first introduced by Jessen and Wintner in [8]. When $\lambda \in \left(0, \frac{1}{2}\right)$, it is well known that μ_λ is singular (see e.g., [10]). When $\lambda = \frac{1}{2}$ it is clear that μ_λ is $\frac{1}{4}$ of the Lebesgue measure on $[-2, 2]$. This means the interesting case is when $\lambda \in \left(\frac{1}{2}, 1\right)$.

Bernoulli convolutions have also been studied by Erdős. In [2] Erdős showed that μ_λ is not absolutely continuous whenever $\lambda^{-1} \in (1, 2)$ is a Pisot number. In his proof he exploited the property of Pisot numbers that powers of Pisot numbers approximate integers exponentially well. These are currently the only values of $\lambda \in \left(\frac{1}{2}, 1\right)$ for which μ_λ is known not to be absolutely continuous.

The typical behavior for Bernoulli convolutions with parameters in $\left(\frac{1}{2}, 1\right)$ is absolute continuity. In [3] by a beautiful combinatorial argument, Erdős showed that there is some $c < 1$ such that for almost all $\lambda \in (c, 1)$, we have that μ_λ is absolutely continuous. This was extended by Solomyak in [17] to show that we may take $c = \frac{1}{2}$. This was later extended by Shmerkin in [15] where he showed that the set of exceptional parameters has Hausdorff dimension 0. These results have been further extended by Shmerkin in [16] who showed that for every $\lambda \in \left(\frac{1}{2}, 1\right)$ apart from an exceptional set of zero Hausdorff dimension μ_λ is absolutely continuous with density in L^q for all finite $q > 1$.

In a ground breaking paper [5], Hochman made progress on a related problem by showing, amongst other things, that if $\lambda \in (\frac{1}{2}, 1)$ is algebraic and not the root of any polynomial with coefficients $0, \pm 1$ then μ_λ has dimension 1. Much of the progress in the last decade builds on the results of Hochman.

There are relatively few known explicit examples of λ for which μ_λ is absolutely continuous. It can easily be shown that for example the Bernoulli convolution with parameter $2^{-\frac{1}{k}}$ is absolutely continuous when k is a positive integer. This is because it may be written as the convolution of the Bernoulli convolution with parameter $\frac{1}{2}$ with another measure. Generalizing this in [4], Garsia showed that if $\lambda \in (\frac{1}{2}, 1)$ has Mahler measure 2, then μ_λ is absolutely continuous. It is worth noting that the condition that λ has Mahler measure 2