ISSN 0012-9593

quatrième série - tome 57 fascicule 4 juillet-août 2024

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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13288 Marseille Cedex 09
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Tarifs

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 57, 2024, p. 1233 à 1270

ON SMOOTHING PROPERTIES AND TAO'S GAUGE TRANSFORM OF THE BENJAMIN-ONO EQUATION ON THE TORUS

BY Patrick GÉRARD, Thomas KAPPELER AND Petar TOPALOV

ABSTRACT. – We prove smoothing properties of the solutions of the Benjamin-Ono equation in the Sobolev space $H^s(\mathbb{T},\mathbb{R})$ for any $s \geq 0$. To this end we show that Tao's gauge transform is a high frequency approximation of the nonlinear Fourier transform for the Benjamin-Ono equation, constructed in our previous work. The results of this paper are manifestations of the quasi-linear character of the Benjamin-Ono equation.

RÉSUMÉ. – Nous établissons des propriétés de régularisation pour les solutions de l'équation de Benjamin-Ono dans l'espace de Sobolev $H^s(\mathbb{T};\mathbb{R})$ pour tout $s \geq 0$. À cette fin nous montrons que la transformation de jauge de Tao est une approximation à haute fréquence de la transformation de Fourier non linéaire pour l'équation de Benjamin-Ono, construite dans notre précédent travail. Les résultats de cet article sont des manifestations du caractère quasi-linéaire de l'équation de Benjamin-Ono.

1. Introduction

In this paper we consider the Benjamin-Ono (BO) equation on the torus,

$$\partial_t v = \mathrm{H}[\partial_x^2 v] - \partial_x v^2, \qquad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},\ t \in \mathbb{R}, \tag{1}$$

where $v \equiv v(t,x)$ is real valued and H denotes the Hilbert transform, defined for $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}$, $\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$, by

$$\mathrm{H}[f](x) := \sum_{n\in\mathbb{Z}} -i\ \mathrm{sign}(n)\widehat{f}(n)\ e^{inx}$$

with $\mathrm{sign}(\pm n) := \pm 1$ for any $n \geq 1$, whereas $\mathrm{sign}(0) := 0$.

Equation (1) has been introduced by Benjamin [2] and Davis & Acrivos [3] to model long, uni-directional internal waves in a two-layer fluid. It has been extensively studied, both on

T.K. was partially supported by the Swiss National Science Foundation and deceased on May 30, 2022. P.T. is partially supported by the Simons Foundation, Award #526907.

 doi:10.24033/asens.2590

the real line $\mathbb{R}$ and on the torus $\mathbb{T}$. Let us briefly summarize some of the by now classical results on the well-posedness problem of (1), relevant for this paper—we refer to [27] for an excellent survey as well as a derivation of (1). Based on work of Saut [26], Abdelouhab & Bona & Felland & Saut proved in [1] that for $s > 3/2$, Equation (1) is globally in time well-posed on the Sobolev space $H^s_r \equiv H^s(\mathbb{T}, \mathbb{R})$ (endowed with the standard norm $\|\cdot\|_s$, defined by (31) below), meaning the following:

(S1) *Existence and uniqueness of classical solutions:* For any initial data $v_0 \in H^s_r$, there exists a unique curve $v : \mathbb{R} \to H^s_r$ in $C(\mathbb{R}, H^s_r) \cap C^1(\mathbb{R}, H^{s-2}_r)$ so that $v(0) = v_0$ and for any $t \in \mathbb{R}$, Equation (1) is satisfied in H^{s-2}_r. (Since H^s_r is an algebra, one has $\partial_x(v(t)^2) \in H^{s-1}_r$ for any time $t \in \mathbb{R}$.)

(S2) *Continuity of solution map:* The solution map $\mathcal{S} : H^s_r \to C(\mathbb{R}, H^s_r)$ is continuous, meaning that for any $v_0 \in H^s_r$, $T > 0$, and $\varepsilon > 0$ there exists $\delta > 0$, so that for any $w_0 \in H^s_r$ with $\|w_0 - v_0\|_s < \delta$, the solutions $w(t) = \mathcal{S}(t, w_0)$ and $v(t) = \mathcal{S}(t, v_0)$ of (1) with initial data $w(0) = w_0$ and, respectively, $v(0) = v_0$ satisfy $\sup_{|t|\le T} \|w(t) - v(t)\|_s \le \varepsilon$.

In the sequel, further progress has been made on the well-posedness of (1) on Sobolev spaces of low regularity. The best results so far in this direction were obtained by Molinet, using as a key ingredient the gauge transform, introduced by Tao [28] for the Benjamin-Ono equation on $\mathbb{R}$. Molinet's results in [23] (cf. also [24]) imply that the solution map $\mathcal{S}$, introduced in $(S2)$ above, continuously extends to any Sobolev space H^s_r with $0 \le s \le 3/2$. More precisely, for any such s, $\mathcal{S} : H^s_r \to C(\mathbb{R}, H^s_r)$ is continuous and for any $v_0 \in H^s_r$, $\mathcal{S}(t, v_0)$ satisfies Equation (1) in H^{s-2}_r. Finally, in the recent paper [12] we proved that (1) is wellposed in the Sobolev space H^s_r for any $s > -1/2$, but illposed for $s \le -1/2$.

In a straightforward way one verifies that for any solution $v(t) \equiv \mathcal{S}(t, v_0)$ of (1) in H^s_r with $s > -1/2$, the mean $\langle v(t)|1\rangle$ is conserved. Here $\langle \cdot | \cdot \rangle$ denotes the extension of the L^2-inner product,

$$\langle f|g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\overline{g}dx, \qquad \forall\, f, g \in L^2_c \equiv L^2(\mathbb{T}, \mathbb{C}) \tag{2}$$

to the dual pairing $H^s_r \times H^{-s}_r \to \mathbb{C}$. As a consequence, for any $s > -1/2$, the subspace

$$H^s_{r,0} := \{v \in H^s_r \;:\; \langle v|1\rangle = 0\}, \tag{3}$$

of H^s_r is invariant by the flow of (1). (For $s = 0$, we usually write $L^2_{r,0}$ for $H^0_{r,0}$.)

Since for any $a \in \mathbb{R}$ and any solution $v(t) = \mathcal{S}(t, v_0)$ of (1) in H^s_r with $s > -1/2$, $v_a(t, x) := a + v(t, x - 2at)$ is again a solution of (1) in H^s_r, for our purposes, it suffices to consider solutions in $H^s_{r,0}$.

The main goal of this paper is to prove smoothing properties of solutions of (1). A first key ingredient in their proof is Tao's gauge transform, which we denote by $\mathcal{G}$. To define it, we first need to introduce some more notation. For any $f \in H^s_c \equiv H^s(\mathbb{T}, \mathbb{C})$, $s \in \mathbb{R}$, the Szegő projection Πf of $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}$ is defined as $\sum_{n\ge 0} \widehat{f}(n)e^{inx}$. Clearly, Π defines a bounded linear operator $H^s_c \to H^s_+$ where

$$H^s_+ := \{f \in H^s_c \;:\; \widehat{f}(n) = 0 \;\; \forall\, n < 0\}.$$

Furthermore, we denote by ∂_x^{-1} the operator

$$\partial_x^{-1} : H_c^s \to H_{c,0}^{s+1},\ f \mapsto \sum_{n\neq 0} \frac{1}{in}\widehat{f}(n)e^{inx},$$

where for any $s \in \mathbb{R}$, $H_{c,0}^s := \{v \in H_c^s : \langle v|1\rangle = 0\}$. For notational convenience, the restriction of ∂_x^{-1} to $H_{c,0}^s$ is also denoted by ∂_x^{-1}.

For our purposes, it suffices to consider solutions of (1) in the Sobolev spaces $H_{r,0}^s$ with $s \geq 0$. For any $u \in H_{r,0}^s$ with $s \geq 0$, we denote by $\mathcal{G}(u)$ (the following version of) Tao's gauge transform of u [28, 24],

$$\mathcal{G}(u) := \partial_x \Pi e^{-i\partial_x^{-1}u}. \tag{4}$$

It was pointed out in [28] that $\mathcal{G}$ can be viewed as a complex version of the Cole-Hopf transform, which was introduced independently by Cole and Hopf in the early fifties to convert Burgers' equation $\partial_t u = \partial_x(\partial_x u - u^2)$ into the heat equation. See e.g., [7, Section 4.4].

Note that

$$\partial_x \Pi[e^{-i\partial_x^{-1}u}] = \Pi[\partial_x e^{-i\partial_x^{-1}u}] = -i\,\Pi[u e^{-i\partial_x^{-1}u}]$$

and that for any $s \geq 0$,

$$\mathcal{G} : H_{r,0}^s \to H_{+,0}^s,\ u \mapsto \partial_x \Pi e^{-i\partial_x^{-1}u},$$

is a real analytic map, where

$$H_{+,0}^s := \{f \in H_+^s : \langle f|1\rangle = 0\}, \qquad H_{+,0} \equiv H_{+,0}^0.$$

It turns out that for any $s \geq 0$, $\mathcal{G}$ is a diffeomorphism onto an open proper subset of $H_{+,0}^s$. See Appendix B for a proof.

Given any initial data $u_0 \in H_{r,0}^s$ with $s \geq 0$, let $u(t) = \mathcal{S}(t, u_0)$ and denote by $w(t) = \mathcal{G}(u(t))$ the gauge transform of $u(t)$, i.e.,

$$w(t) = \partial_x \Pi[e^{-i\partial_x^{-1}u(t)}].$$

For notational convenience, we will often not explicitly indicate the dependence of u, v, and w on t in the sequel. Let us derive the equation, satisfied by $w(t)$. Since $\partial_x^{-1}\partial_x(u^2) = u^2 - \langle u^2|1\rangle$ one sees that $v(t) := \partial_x^{-1}u(t)$ satisfies

$$\partial_t v = \mathrm{H}[\partial_x^2 v] - (\partial_x v)^2 + \langle(\partial_x v)^2|1\rangle, \qquad v(0) = \partial_x^{-1}u_0. \tag{5}$$

Furthermore, using that $\partial_t w = \partial_x \Pi[-i\partial_t v \cdot e^{-iv}]$ and

$$\partial_x^2 w = \partial_x \Pi[-i\partial_x^2 v \cdot e^{-iv} - (\partial_x v)^2 e^{-iv}],$$

one computes

$$\partial_t w + i\partial_x^2 w = \partial_x \Pi[-i\partial_t v \cdot e^{-iv} + \partial_x^2 v \cdot e^{-iv} - i(\partial_x v)^2 e^{-iv}].$$

Since for any $f \in H_c^s$, the Hilbert transform $\mathrm{H}[f]$ of f satisfies $\mathrm{H}[f] = -if + 2i(\mathrm{Id} - \Pi)[f]$ one infers that

$$\partial_x^2 v = i\mathrm{H}[\partial_x^2 v] + 2(\mathrm{Id} - \Pi)[\partial_x^2 v].$$

Combining the latter identity with (5) then yields

$$\partial_t w + i\partial_x^2 w = \partial_x \Pi\big[-i\langle(\partial_x v)^2|1\rangle e^{-iv} + 2(e^{-iv} \cdot (\mathrm{Id} - \Pi)(\partial_x^2 v)\big].$$