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NON-ARCHIMEDEAN QUANTUM K-INVARIANTS

BY Mauro PORTA AND Tony Yue YU

Abstract. – We construct quantum K-invariants in non-archimedean analytic geometry. Contrary to the classical approach in algebraic geometry via perfect obstruction theory, we build on our previous works on the foundations of derived non-archimedean geometry, the representability theorem and Gromov compactness. We obtain a list of natural geometric relations between the stacks of stable maps, directly at the derived level, with respect to elementary operations on graphs. They imply immediately the corresponding properties of quantum K-invariants. The derived approach produces highly intuitive statements and functorial proofs. The flexibility of our derived approach to quantum K-invariants allows us to impose not only simple incidence conditions for marked points, but also incidence conditions with multiplicities. This leads to a new set of enumerative invariants. Our motivations come from non-archimedean enumerative geometry and mirror symmetry.

Résumé. – Nous construisons des K-invariants quantiques en géométrie analytique non archimédienne. Contrairement à l'approche classique en géométrie algébrique via les théories d'obstruction parfaites, nous nous appuyons sur nos travaux précédents sur les fondements de la géométrie dérivée non archimédienne, le théorème de représentabilité et la compacité de Gromov. Nous obtenons une liste des relations géométriques naturelles des champs d'applications stables, directement au niveau dérivé, par rapport aux opérations élémentaires sur les graphes. Cela implique immédiatement les propriétés correspondantes des K-invariants quantiques. L'approche dérivée produit des énoncés intuitifs et des preuves fonctorielles. La flexibilité de notre approche dérivée nous permet d'imposer non seulement des conditions d'incidence simples pour des points marqués, mais aussi des conditions d'incidence avec des multiplicités. Cela donne lieu à de nouveaux invariants énumératifs. Nos motivations viennent de la géométrie énumérative non archimédienne et de la symétrie miroir.

1. Introduction

Let X be a smooth projective complex variety. Its system of quantum K-invariants is the collection of linear maps

$$K^X_{g,n,\beta}\colon K_0(X)^{\otimes n} \longrightarrow K_0(\overline{M}_{g,n}),$$

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for all $g, n \in \mathbb{N}$ and $\beta \in H_2(X)$, defined by

$$K^X_{g,n,\beta}(a_1 \otimes \cdots \otimes a_n) := \mathrm{st}_*\big(\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)} \otimes \mathrm{ev}_1^* a_1 \otimes \cdots \otimes \mathrm{ev}_n^* a_n\big),$$

where $\overline{M}_{g,n}(X, \beta)$ is the moduli stack of n-pointed genus g stable maps into X of class β, $\mathcal{O}^{\mathrm{vir}}$ is its virtual structure sheaf, st is the stabilization of domain, and ev_i are the evaluation maps (see [40]). Coupling with classes in $K_0(\overline{M}_{g,n})$, we obtain numerical invariants that are non-negative integers, which manifests (in a subtle way) the geometry of curves in X.

Quantum K-invariants were first studied by Givental [21] in the case $g = 0$ and X is convex. In this case the moduli stack $\overline{M}_{g,n}(X, \beta)$ is smooth with expected dimension, the virtual structure sheaf is thus equal to the structure sheaf. The virtual structure sheaves in the general case are constructed by Lee [40], using the techniques of perfect obstruction theory and intrinsic normal cone developed by Behrend-Fantechi [9]. We refer to [41, 24, 30, 22, 59] for further developments of quantum K-theory in algebraic geometry, and to [23, 13, 52] for more specific examples and applications.

Motivated by the non-archimedean approach to mirror symmetry (see [38, 39, 50]), in this paper we construct quantum K-invariants in non-archimedean analytic geometry, more precisely, for proper smooth rigid analytic spaces over any discrete valuation field k of residue characteristic zero. Non-archimedean geometry comes into play naturally as we study degenerations of Calabi-Yau varieties in mirror symmetry. It has special features in comparison with classical approaches by symplectic geometry and complex geometry. In particular, we now have a rigorous foundation of non-archimedean SYZ (Strominger-Yau-Zaslow) fibration (see [50]), whose original archimedean version is yet far beyond reach. More importantly, the enumerative invariants responsible for instanton corrections in the SYZ conjecture [61] cannot be well-defined in the archimedean setting; while the non-archimedean enumerative invariants are more promising candidates, as already demonstrated in special cases of log Calabi-Yau varieties in [66, 68, 31, 26]. Via formal models of non-archimedean analytic spaces, non-archimedean enumerative invariants are intimately related to logarithmic enumerative invariants of the special fiber (see the works of Abramovich, Chen, Gross, Siebert [15, 2, 25, 3]).

Our approach to the construction of non-archimedean quantum K-invariants differs from the classical one in algebraic geometry via perfect obstruction theory. Instead, we build upon our previous works on the foundations of derived non-archimedean geometry [55, 56, 57, 58]. The results of this paper are supposed to bring powerful techniques from derived geometry into the study of mirror symmetry quantum corrections via non-archimedean enumerative methods along the lines of [66, 68, 31, 26].

Now let us sketch the results of this paper. Recall that in order to construct any satisfactory theory of enumerative invariants, there are two main issues to be solved: compactness and transversality (see the excellent expository paper [53] on the two issues for various enumerative theories). The same applies to non-archimedean quantum K-invariants.

In this case, the issue of compactness is solved in [67] via non-archimedean Kähler structures and formal models of non-archimedean stable maps, referred to as the non-archimedean Gromov compactness theorem, which we will review in Section 7. We treat the issue of transversality in the current paper using derived geometry as explained below.

Fix a smooth rigid k-analytic space X.

THEOREM 1.1 (see Theorems 4.13 and 4.15). – *The moduli stack* $\overline{M}_{g,n}(X)$ *of* n*-pointed genus* g *stable maps into* X *admits a natural derived enhancement* $\mathbb{R}\overline{M}_{g,n}(X)$ *that is a derived* k*-analytic stack locally of finite presentation and derived lci.*

Now assuming X proper and endowed with a Kähler structure, combining with the non-archimedean Gromov compactness theorem (Theorem 7.6), we can immediately obtain the non-archimedean quantum K-invariants

$$\begin{aligned} K^X_{g,n,\beta}\colon K_0(X)^{\otimes n} &\longrightarrow K_0(\overline{M}_{g,n}) \\ a_1 \otimes \cdots \otimes a_n &\longmapsto \mathrm{st}_*\big(\mathrm{ev}_1^* a_1 \otimes \cdots \otimes \mathrm{ev}_n^* a_n\big), \end{aligned}$$

where the maps st and ev_i all depart from the derived moduli stack $\mathbb{R}\overline{M}_{g,n}(X,\beta)$.

Next we prove that the non-archimedean quantum K-invariants satisfy all the expected properties analogous to the algebraic case. Instead of manipulating perfect obstruction theories as in [40, 8], our proofs of the properties of the invariants will follow directly from a list of natural and intuitive geometric relations between the derived moduli stacks, which we explain below.

In order to state the geometric relations, instead of working with n-pointed genus g stable maps, one has to work with a slight combinatorial refinement called (τ,β)-marked stable maps for an A-graph (τ,β), introduced by Behrend-Manin [10] for the study of Gromov-Witten invariants. The A-graph (τ,β) imposes degeneration types on the domains of stable maps as well as more refined curve classes (see Section 4.3). We have the associated moduli stack of (τ,β)-marked stable maps, the corresponding derived enhancement, and thus the (more refined) non-archimedean quantum K-invariants $K^X_{\tau,\beta}$.

THEOREM 1.2 (see Theorems 5.1, 5.2, 5.4, 5.7 and 5.19). – *Let* S *be a rigid* k*-analytic space and* X *a rigid* k*-analytic space smooth over* S*. The derived moduli stack* $\mathbb{R}\overline{M}(X/S,\tau,\beta)$ *of* (τ,β)*-marked stable maps into* X/S *associated to an A-graph* (τ,β) *satisfies the following geometric relations with respect to elementary operations on A-graphs:*

(1) Products: *Let* (τ_1,β_1) *and* (τ_2,β_2) *be two A-graphs. We have a canonical equivalence*

$$\mathbb{R}\overline{M}(X/S,\tau_1\sqcup\tau_2,\beta_1\sqcup\beta_2) \xrightarrow{\sim} \mathbb{R}\overline{M}(X/S,\tau_1,\beta_1)\times_S \mathbb{R}\overline{M}(X/S,\tau_2,\beta_2),$$

where the projections are given by the natural forgetful maps.

(2) Cutting edges: *Let* (σ,β) *be an A-graph obtained from* (τ,β) *by cutting an edge* e *of* τ*. Let* i, j *be the two tails of* σ *created by the cut. We have a derived pullback diagram*

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S,\tau,\beta) & \xrightarrow{c} & \mathbb{R}\overline{M}(X/S,\sigma,\beta) \\ \big\downarrow{\scriptstyle \mathrm{ev}_e} & & \big\downarrow{\scriptstyle \mathrm{ev}_i\times\mathrm{ev}_j} \\ X & \xrightarrow{\Delta} & X\times_S X, \end{array}$$

where Δ *is the diagonal map,* ev_e *is evaluation at the section* s_e *corresponding to the edge* e*, and* c *is induced by cutting the domain curves at* s_e*.*