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BOUNDEDNESS OF FANO TYPE FIBRATIONS

BY CAUCHER BIRKAR

ABSTRACT. – In this paper, we prove various results on boundedness and singularities of Fano fibrations and of Fano type fibrations. A Fano fibration is a projective morphism $X \to Z$ of algebraic varieties with connected fibers such that X is Fano over Z, that is, X has "good" singularities and $-K_X$ is ample over Z. A Fano type fibration is similarly defined where X is assumed to be close to being Fano over Z. This class includes many central ingredients of birational geometry such as Fano varieties, Mori fiber spaces, flipping and divisorial contractions, crepant models, germs of singularities, etc. We develop the theory in the more general framework of log Calabi-Yau fibrations.

RÉSUMÉ. – Dans cet article, nous prouvons divers résultats sur les limites et les singularités de fibrations de Fano et les fibrations de type Fano. Une fibration de Fano est un morphisme projectif $X \to Z$ de variétés algébriques à fibres connexes tel que X est Fano sur Z, c'est-à-dire que X a de « bonnes » singularités et $-K_X$ est ample sur Z. Une fibration de type Fano est définie de façon similaire quand X est supposé être proche d'être Fano sur Z. Cette classe comprend de nombreux ingrédients centraux de géométrie birationnelle tels que les variétés de fano, les espaces de fibres Mori, le flip et les contractions divisorielles, les modèles répétiteurs, les germes de singularités, etc. Nous développons la théorie dans le cadre plus général des log-fibrations de Calabi-Yau.

1. Introduction

We work over a fixed algebraically closed field k of characteristic zero unless stated otherwise. Varieties are assumed to be irreducible.

According to the minimal model program (including the abundance conjecture), every variety W is expected to be birational to a projective variety X with good singularities such that either

— X is canonically polarized (i.e., K_X is ample), or

— X admits a Mori-Fano fibration $X \to Z$ (i.e., K_X is anti-ample over Z), or

— X admits a Calabi-Yau fibration $X \to Z$ (i.e., K_X is numerically trivial over Z).

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This reduces the birational classification of algebraic varieties to classifying such X. From the point of view of moduli theory, it makes perfect sense to focus on such X, as they have a better chance of having a reasonable moduli theory, due to the special geometric structures they carry. For this and other reasons, Fano and Calabi-Yau varieties and their fibrations are central to birational geometry. They are also of great importance in many other parts of mathematics such as arithmetic geometry, differential geometry, mirror symmetry, and mathematical physics.

Boundedness properties of canonically polarized varieties and Fano varieties have been extensively studied in the literature leading to recent advances [18, 4, 5]. With the above philosophy of the minimal model program in mind, there is a natural urge to extend such studies to Fano and Calabi-Yau fibrations. Such fibrations also frequently appear in inductive arguments.

In this paper, a *Fano fibration* means a projective morphism $X \to Z$ of algebraic varieties with connected fibers such that X is Fano over Z, that is, X has klt singularities and $-K_X$ is ample over Z (for definition of klt and lc singularities, see 3.8). The following are general guiding questions which are the focus of this paper:

(1) *Under what conditions do Fano fibrations form bounded families?*

(2) *How do singularities behave on the total space and base of a Fano fibration?*

(3) *When do bounded (klt or lc) complements exist for Fano fibrations?*

More precise formulations of these questions include many important and hard problems. When Z is just a point, these questions were studied in [5, 4]. In this paper we are mostly interested in the case $\dim Z > 0$ which poses new challenges that do not appear in the case $\dim Z = 0$.

One of our main results concerns Question (1). Recall that the BAB conjecture states that Fano varieties of given dimension d and with ε-lc singularities form a bounded family, where $\varepsilon > 0$. We would like to extend this result to the relative setting. Consider Fano fibrations $f: X \to Z$ where Z is projective and varies in a bounded family. What kind of conditions should be imposed to ensure that the X form a bounded family? Let us assume that X has dimension d and has ε-lc singularities, with $\varepsilon > 0$. It turns out that this is not enough even when $Z = \mathbb{P}^1$ and X is a smooth surface, see Example 2.1. To get boundedness, we need more subtle conditions. A special case of our main results says that if we take a very ample divisor A on Z with bounded $A^{\dim Z}$ so that $f^*A - K_X$ is nef, then we can ensure X varies in a bounded family. For more general statements see Theorems 1.2, 1.3, 2.3.

In order to prove Theorems 1.2 and 1.3, we will need to treat variants of Questions (2) and (3). Indeed, we find bounded natural numbers m, n and produce a boundary divisor Λ so that $n(K_X + \Lambda) \sim mf^*A$ (see Theorem 1.7). This is a kind of bounded klt complement as in Question (3), so we are led deep into the theory of complements. To construct this complement, in turn we need to control singularities which is related to Question (2) (see Theorems 1.4, 1.6).

In our definition of Fano fibrations, we also allow f to be birational. In this case, we get boundedness of crepant models which appears frequently in applications. This is explained in more details below.

The results of this paper are motivated by the classification theory of algebraic varieties. Theorem 1.2 can be considered as the first step towards the classification of Fano fibrations (see the beginning of Section 2). On the other hand, such a boundedness statement is often required to perform inductive proofs and to deduce boundedness properties from birational models. In fact, Theorem 1.2 and other results of this paper have found many important applications. Here we recall a partial list:

— boundedness of polarized varieties [6],

— boundedness and volumes of generalized pairs which are in turn applied to questions on varieties of intermediate Kodaira dimension and stable minimal models [7],

— boundedness of elliptic Calabi-Yau varieties [9] (also see [12]),

— boundedness of rationally connected 3-folds X with $-K_X$ nef [9, 11].

In the rest of this introduction, we state our main results. We will use the language of log Calabi-Yau fibrations. In the next section, we state more general results in the context of generalized pairs.

Log Calabi-Yau and Fano type fibrations

Now we introduce the notion which unifies many central ingredients of birational geometry. A *log Calabi-Yau fibration* consists of a pair (X, B) with log canonical singularities and a contraction $f: X \to Z$ (i.e., a surjective projective morphism with connected fibers) such that $K_X + B \sim_{\mathbb{R}} 0$ relatively over Z. We usually denote the fibration by $(X, B) \to Z$. Note that we allow the two extreme cases: when f is birational and when f is constant. When f is birational, such a fibration is a crepant model of (Z, f_*B) (see below). When f is constant, that is, when Z is a point, we just say (X, B) is a *log Calabi-Yau pair*. In general, if F is a general fiber of f and if we let $K_F + B_F = (K_X + B)|_F$, then $K_F + B_F \sim_{\mathbb{R}} 0$, hence (F, B_F) is a log Calabi-Yau pair justifying the terminology.

The class of log Calabi-Yau fibrations includes all log Fano and log Calabi-Yau varieties and much more. For example, if X is a variety which is Fano over a base Z, then we can easily find B so that $(X, B) \to Z$ is a log Calabi-Yau fibration. This includes all Mori fiber spaces. Since we allow birational contractions, it also includes all divisorial and flipping contractions. Another interesting example of log Calabi-Yau fibrations $(X, B) \to Z$ is when $X \to Z$ is the identity morphism; the set of such fibrations simply coincides with the set of pairs with log canonical singularities. On the other hand, a surface with a minimal elliptic fibration over a curve is another instance of a log Calabi-Yau fibration.

A log Calabi-Yau fibration $(X, B) \to Z$ is of *Fano type* if X is of Fano type over Z, that is, if $-(K_X + C)$ is ample over Z and (X, C) is klt for some boundary C. When (X, B) is klt, this is equivalent to saying that $-K_X$ is big over Z.

We introduce some notation, somewhat similar to [20], to simplify the statements of our results below.

Definition 1.1. – *Let d, r be natural numbers and ε a positive real number. A (d, r, ε)*-Fano type (log Calabi-Yau) fibration *consists of a pair (X, B) and a contraction $f: X \to Z$ such that we have the following:*

— *(X, B) is a projective ε-lc pair of dimension d,*