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WEIGHT CONJECTURES FOR ℓ-COMPACT GROUPS AND SPETSES

BY Radha KESSAR, Gunter MALLE AND Jason SEMERARO

Abstract. – Fundamental conjectures in modular representation theory of finite groups, more precisely, Alperin's weight conjecture and Robinson's ordinary weight conjecture, can be expressed in terms of fusion systems. We use fusion systems to connect the modular representation theory of finite groups of Lie type to the theory of ℓ-compact groups. Under some mild conditions we prove that the fusion systems associated to homotopy fixed points of ℓ-compact groups satisfy an equation which for finite groups of Lie type is equivalent to Alperin's weight conjecture.

For finite reductive groups, Robinson's Ordinary weight conjecture is closely related to Lusztig's Jordan decomposition of characters and the corresponding results for Brauer ℓ-blocks. Motivated by this, we define the principal block of a spets attached to a spetsial $\mathbb{Z}_\ell$-reflection group, using the fusion system related to it via ℓ-compact groups, and formulate an analogue of Robinson's conjecture for this block. We prove this formulation for an infinite family of cases as well as for some groups of exceptional type.

Our results not only provide further strong evidence for the validity of the weight conjectures, but also point toward some yet unknown structural explanation for them purely in the framework of fusion systems.

Résumé. – Des conjectures fondamentales en théorie des représentations modulaires des groupes finis, plus précisément, la conjecture d'Alperin et la conjecture des poids ordinaires de Robinson, peuvent être exprimées en termes de systèmes de fusion. Nous utilisons les systèmes de fusion pour relier la théorie des représentations modulaires des groupes finis de type Lie à la théorie des groupes ℓ-compacts. Sous des conditions faibles, nous prouvons que le système de fusion associé aux points fixes d'homotopie des groupes ℓ-compacts satisfait à une équation qui, pour les groupes finis de type de Lie, est équivalente à la conjecture d'Alperin.

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Pour les groupes réductifs finis, la conjecture des poids ordinaires de Robinson est étroitement liée à la décomposition des caractères de Jordan demontrée par Lusztig et aux résultats sur les ℓ-blocs de Brauer. Motivés par cela, nous définissons le bloc principal d'un spets attaché à un groupe de $\mathbb{Z}_\ell$-reflexions dit spetsial, en utilisant le système de fusion qui lui est associé via les groupes ℓ-compacts, et formulons un analogue de la conjecture de Robinson pour ce bloc. Nous prouvons cette formulation pour une famille infinie de cas ainsi que pour certains groupes de type exceptionnel.

Nos résultats fournissent non seulement d'autres indications de la validité de la conjecture de poids, mais pointent également vers certaines explications structurales encore inconnues purement dans le cadre des systèmes de fusion.

1. Introduction and statement of results

In this paper we connect fusion systems originating from homotopy fixed points on ℓ-compact groups with the representation theory of so-called spetses by means of a natural generalization of Alperin's weight conjecture from modular representation theory of finite groups.

Let ℓ be a prime. An *ℓ-compact group* is a topological object that may loosely be regarded as a homotopical analogue of a compact Lie group, built from an ℓ-adic reflection group as Weyl group. It has been shown to give rise to fusion systems on certain finite ℓ-groups. In representation theory, a *spets* is a yet rather mysterious analogue of a finite reductive group with a complex reflection group as Weyl group, for which some shadow of Deligne-Lusztig character theory can be formulated. Apparent similarities between these two classes of objects have led a number of authors to expect that there should be a more formal connection between them. With this goal in mind, the third author [52] observed various numerical consistencies between the exotic 2-fusion systems associated to a 2-compact group with Weyl group the exceptional complex reflection group G_{24} and an ad hoc defined set of irreducible characters associated to the spets with the same Weyl group, in the spirit of a generalization of Alperin's weight conjecture. This famous local-global conjecture from 1986 is an attempt to relate the character theory of a finite group to the fusion system on its Sylow subgroups.

In the present paper, we provide a formal theoretical context to these observations. Thus the goals of the present paper are to:

- associate various global invariants to spetses (such as the principal ℓ-block);
- formulate analogues for spetses and ℓ-compact groups of various local-global counting conjectures for groups;
- prove these conjectures (in some cases);
- highlight/explain techniques from algebraic topology relevant to the study of group representations.

In order to motivate and explain our constructions, conjectures and results, we begin by discussing global and local data associated to finite groups of Lie type.

A *finite group of Lie type* is the group of fixed points $\mathbf{G}^F$ of a connected reductive linear algebraic group $\mathbf{G}$ under a Steinberg endomorphism F with respect to an $\mathbb{F}_q$-structure for some prime power q coprime to ℓ. The ordinary irreducible characters of $\mathbf{G}^F$ are constructed from the ℓ-adic cohomology groups of so-called Deligne-Lusztig varieties on which $\mathbf{G}^F$ acts. Lusztig has given a combinatorial parametrisation of these characters by *Lusztig series* $\mathcal{E}(G, s)$ indexed by (classes of) semisimple elements s in the Langlands dual group, which is purely in terms of the Weyl group W of $\mathbf{G}$. Furthermore, for any prime ℓ different from the defining characteristic of $\mathbf{G}$, the distribution of these characters into Brauer ℓ-blocks has also been shown to be controlled by W. This is described by *e-Harish-Chandra theory*, where e denotes the order of q modulo ℓ. For example, when ℓ is very good for $\mathbf{G}$ then the unipotent characters of $\mathbf{G}^F$ lying in the principal ℓ-block B_0 are those in the *principal* e-Harish-Chandra series [17]. Moreover, the latter is in bijection with the irreducible characters of the corresponding relative Weyl group denoted here W_e, a complex reflection group provided by Lehrer-Springer theory.

The local data we consider for $\mathbf{G}^F$ is most succinctly described in terms of fusion systems. Recall that a *fusion system $\mathcal{F}$ on a finite ℓ-group S* is a category with objects the subgroups of S and morphisms certain injective group homomorphisms satisfying some weak axioms. It is called *saturated* if it satisfies two additional “Sylow axioms”, derived from the motivating example of the fusion system $\mathcal{F}_\ell(G)$ induced by a finite group G on a Sylow ℓ-subgroup. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(Q) = Z(Q)$ for all $\mathcal{F}$-conjugates Q of P and *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ has no non-trivial normal ℓ-subgroup. The class $\mathcal{F}^{\mathrm{cr}}$ of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups plays an important role in the local-global conjectures we consider. If k is an algebraically closed field of characteristic ℓ, G is a finite group, B_0 is the principal ℓ-block of G and $\mathrm{IBr}(B_0)$ the set of irreducible ℓ-Brauer characters of G lying in B_0, *Alperin's weight conjecture* (AW conjecture) [2] claims the equality $|\mathrm{IBr}(B_0)| = \mathbf{w}(\mathcal{F}_\ell(G))$ where, for a saturated fusion system $\mathcal{F}$,

$$\mathbf{w}(\mathcal{F}) := \sum_{P \in \mathcal{F}^{\mathrm{cr}}/\mathcal{F}} z(k\mathrm{Out}_{\mathcal{F}}(P)),$$

and the sum runs over $\mathcal{F}$-conjugacy class representatives of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups. Here, for a finite group H, $z(kH)$ denotes the number of isomorphism classes of projective simple kH modules.

In the case $\mathbf{G}^F$ is a finite group of Lie type with Weyl group W then assuming ℓ is very good for $\mathbf{G}$, and q of order e modulo ℓ, then $|\mathrm{IBr}(B_0)|$ equals $|\mathrm{Irr}(W_e)|$ (see Proposition 4.1). Hence, the above discussion implies that the AW conjecture for the principal ℓ-block is equivalent to the assertion

$$\mathbf{w}(\mathcal{F}_\ell(\mathbf{G}^F)) = |\mathrm{Irr}(W_e)|\,. \tag{1}$$

We wish to argue that the left hand side of this equality is also “generic” in the Weyl group W. For this, we require some algebraic topology, specifically the theory of ℓ-compact groups.

To any ℓ-compact group X is associated a reflection group W on a $\mathbb{Z}_\ell$-lattice L. If X is connected, then any prime power q coprime to ℓ determines a self-equivalence ψ^q of X, unique up to homotopy, called an unstable Adams operator. If X is moreover simply