ISSN 0012-9593

*quatrième série - tome 57 fascicule 3 mai-juin 2024*

*ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE*

Thomas J. HAINES, João LOURENÇO & Timo RICHARZ

# *On the normality of Schubert varieties: Remaining cases in positive characteristic*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique



**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

**Édition et abonnements / *Publication and subscriptions***

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

**Tarifs**

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.



ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4[e] série, t. 57, 2024, p. 895 à 959

# ON THE NORMALITY OF SCHUBERT VARIETIES: REMAINING CASES IN POSITIVE CHARACTERISTIC

BY Thomas J. HAINES, João LOURENÇO AND Timo RICHARZ

ABSTRACT. – We study the geometry of equicharacteristic partial affine flag varieties associated to tamely ramified groups $G$, with particular attention to the characteristic $p > 0$ setting. We prove that when $p$ divides the order of the fundamental group $\pi_1(G_{\mathrm{der}})$, most Schubert varieties attached to $G$ are not normal, and we provide a criterion for when normality holds. Apart from this, we show, on the one hand, that loop groups of semisimple groups satisfying $p \mid \#\pi_1(G_{\mathrm{der}})$ are not reduced, and on the other hand, that their integral realizations are ind-flat. Our methods allow us to classify all tamely ramified Pappas-Zhu local models of Hodge type which are normal.

RÉSUMÉ. – Nous étudions la géométrie des variétés de drapeaux affines partielles associées à des groupes $G$ modérément ramifiés, avec un accent particulier sur le cadre de la caractéristique $p > 0$. On démontre que, lorsque $p$ divise l'ordre du groupe fondamental $\pi_1(G_{\mathrm{der}})$, la plupart des variétés de Schubert ne sont pas normales et nous fournissons une condition nécessaire et suffisante pour que cela se produise. De plus, nous montrons, d'une part, que les groupes de lacets de groupes semisimples satisfaisant $p \mid \#\pi_1(G_{\mathrm{der}})$ ne sont pas réduits, et d'autre part, que leurs réalisations intégrales sont ind-plates. Nos méthodes nous permettent de classifier tous les modèles locaux de type Hodge au sens de Pappas-Zhu qui sont normaux.

## 1. Introduction

Partial affine flag varieties are important objects in arithmetic algebraic geometry for their intimate relation to local models of Shimura varieties and moduli stacks of shtukas. They first appeared extensively in the realm of Kac-Moody theory by means of (integral) representation theory of Kac-Moody algebras. They were later reinterpreted via the theory

Research of T.H. partially supported by NSF DMS-1801352, research of J.L. funded by the Deutsche Forschungsgemeinschaft and research of T.R. funded by the European Research Council (ERC) under Horizon Europe (grant agreement nº 101040935) and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) TRR 326 *Geometry and Arithmetic of Uniformized Structures*, project number 444845124.

 doi:10.24033/asens.2584

of affine Grassmannians as parametrizing torsors under parahoric group schemes over the formal disk equipped with a trivialization over the punctured one.

In the works of Faltings [19], Pappas-Rapoport [53], Zhu [71] and Pappas-Zhu [55], the authors establish several geometric properties of affine flag varieties, such as normality of Schubert varieties or reducedness of the special fiber of local models, under the following working hypothesis: the reductive group $G$ over the non-archimedean local base field is *tamely ramified*, and the residue characteristic $p > 0$ *does not divide* the order of $\pi_1(G_{\mathrm{der}})$, that is, the simply connected cover $G_{\mathrm{sc}} \to G_{\mathrm{der}}$ of the derived subgroup of $G$ is an étale isogeny. The first type of restriction has been substantially lifted in the work of Levin [43] for Weil-restricted groups, and in [47] for absolutely almost simple, wildly ramified groups. The second type of restriction is dealt with in this paper, whose main finding can be summarized as follows: Let $F = k((t))$ be the Laurent series field in the formal variable $t$ with algebraically closed residue field $k$ of characteristic $p > 0$. Let $G$ be a tamely ramified connected reductive $F$-group, $\mathbf{f}$ a facet of its Bruhat-Tits building and $\mathbf{a}$ an alcove containing $\mathbf{f}$ in its closure. For each class $w \in W/W_{\mathbf{f}}$ in the Iwahori-Weyl group quotient, let $S_w = S_w(\mathbf{a}, \mathbf{f})$ be the associated Schubert variety in the partial affine flag variety $\mathrm{Fl}_{G,\mathbf{f}}$. Note that $W/W_{\mathbf{f}}$ is always a (countable) infinite set when $G$ is nontrivial.

THEOREM 1.1 (Prop. 2.1, Thm. 2.5, Prop. 6.5, App. D). – *Assume $G$ is absolutely almost simple* (*in particular, semisimple*). *If $p$ divides $\#\pi_1(G)$, then only finitely many Schubert varieties $S_w$, $w \in W/W_{\mathbf{f}}$ in the partial affine flag variety $\mathrm{Fl}_{G,\mathbf{f}}$ are normal. The non-normal Schubert varieties are geometrically unibranch and regular in codimension* 1, *but do not satisfy the* ($S2$) *property, do not have rational singularities, and are neither Cohen-Macaulay, nor weakly normal, nor Frobenius split.*

The existence of non-normal Schubert varieties in bad residue characteristics was first observed by the second named author. This came as a total surprise to us as these seem to be the very first examples of non-normal Schubert varieties in the literature. The easiest such example occurs for the quasi-minuscule Schubert variety inside the affine Grassmannian for $G = \mathrm{PGL}_2$ in residue characteristic $p = 2$: the complete local ring at the singular point is isomorphic to the $k$-algebra

$$k[[x, y, v, w]]/(vw + x^2y^2, v^2 + x^3y, w^2 + xy^3, xw + yv).$$

This is a surface singularity which is not weakly normal. Its (weak) normalization morphism identifies with the inclusion map of the subalgebra of $k[[x, y, z]]/(z^2 + xy)$ generated by $x, y, v = xz, w = yz$ (see Appendix B).

The reason why non-normal Schubert varieties must exist can be summarized in a few lines. Up to translation by a suitable element in $G(F)$ which stabilizes $\mathbf{a}$, we may assume that $S_w$ lies in the neutral component of $\mathrm{Fl}_{G,\mathbf{f}}$. By [53, Prop. 6.6], the reduction of the neutral component identifies with that of $G_{\mathrm{der}}$, so for this discussion we may assume that $G = G_{\mathrm{der}}$ is semisimple. Then one has a map

$$S_{\mathrm{sc},w} = S_{\mathrm{sc},w}(\mathbf{a}, \mathbf{f}) \longrightarrow S_w(\mathbf{a}, \mathbf{f}) = S_w \tag{1.1}$$

where $S_{\mathrm{sc},w}$ is the Schubert variety for $w$ inside $\mathrm{Fl}_{G_{\mathrm{sc}},\mathbf{f}}$ and $G_{\mathrm{sc}} \to G$ is the simply connected cover. The Schubert variety $S_{\mathrm{sc},w}$ is known to be normal by [53, Thm. 8.4], and the map (1.1) can be shown to be finite, birational and a universal homeomorphism by using Demazure resolutions. In other words, the map (1.1) is the (weak) normalization morphism of $S_w$, just as in the example of the quasi-minuscule Schubert variety above. But the affine flag variety $\mathrm{Fl}_{G_{\mathrm{sc}},\mathbf{f}}$ is reduced as an ind-scheme by [53, Thm. 6.1], that is, equals the colimit of its Schubert varieties. If all Schubert varieties in $\mathrm{Fl}_{G,\mathbf{f}}$ were normal, then these two facts would imply the map $\mathrm{Fl}_{G_{\mathrm{sc}},\mathbf{f}} \to \mathrm{Fl}_{G,\mathbf{f}}$ is a monomorphism. By looking at tangent spaces, this is clearly not true as soon as the kernel of $G_{\mathrm{sc}} \to G$ is non-étale, or equivalently, as soon as $p$ divides $\#\pi_1(G)$. We should however stress that the above reasoning only shows that there are infinitely many non-normal Schubert varieties in $\mathrm{Fl}_{G,\mathbf{f}}$ when $p$ divides $\#\pi_1(G)$.

For the rest of this introduction, we let $G$ denote an arbitrary tamely ramified connected reductive $F$-group. Exploiting tangent spaces a bit further, we show that the normality of $S_w$ is equivalent to the injectivity of the induced map $T_e S_{\mathrm{sc},w} \to T_e S_w$ on tangent spaces, which yields the following key observation:

LEMMA 1.2 (Cor. 2.2). – *Let $w \in W/W_{\mathbf{f}}$. If $S_w$ is normal, then $S_v$ is normal for all $v \leq w$.*

In order to give an effective normality criterion, we are led to a deeper study of tangent spaces of Schubert varieties for simply connected groups. In this, we recast in Section 4 results of Kumar [41], Mathieu [50], Ramanathan [58] and Polo [56] in the following fashion.

We lift our whole setting to the Witt vectors $\mathbb{W}(k)$ as in [53, §§7–9], and denote by $\underline{S}_{\mathrm{sc},w} \subset \underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}$ the lift to $\mathbb{W}(k)$ of $S_{\mathrm{sc},w} \subset \mathrm{Fl}_{G_{\mathrm{sc}},\mathbf{f}}$ which comes equipped with a section $e\colon \operatorname{Spec} \mathbb{W}(k) \to \underline{S}_{\mathrm{sc},w}$ given by the base point. Given any equivariant ample line bundle $\mathcal{L}$ on $\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}$, we obtain the Kac-Moody action of $T_e\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}$ on the vector space $\Gamma(\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}, \mathcal{L})^\vee$ dual to $\Gamma(\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}, \mathcal{L})$; see Section 5.3.

THEOREM 1.3 (Lem. 5.9). – *Assume $w \in W_{\mathrm{aff}}$. For any $\mathbb{W}(k)$-algebra $R$, the $R$-valued tangent space*

$$T_e\underline{S}_{\mathrm{sc},w}(R) = \mathrm{Hom}_{\mathbb{W}(k)}(e^*\Omega_{\underline{S}_{\mathrm{sc},w}/\mathbb{W}(k)}, R)$$

*identifies with the submodule of $T_e\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}(R)$ consisting of those $X$ such that $X\Theta_{\mathcal{L}}^\vee$ lies in the subspace $\Gamma(\underline{S}_{\mathrm{sc},w}, \mathcal{L})^\vee \otimes R$, where $\Theta_{\mathcal{L}} \in \Gamma(\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}, \mathcal{L})$ is the usual theta divisor attached to $\mathcal{L}$, and $\Theta_{\mathcal{L}}^\vee \in \Gamma(\underline{\mathrm{Fl}}_{G_{\mathrm{sc}},\mathbf{f}}, \mathcal{L})^\vee$ is the vector that sends $\Theta_{\mathcal{L}}$ to 1 and the remaining weight spaces to 0.*

This formula can in principle be used to determine whether a given Schubert variety is normal or not (see Corollary 5.12). We also think that it is of independent interest to have a good source for this material (some of which was known before in related contexts), and that having a Witt-vector framework which links to characteristic zero settings would potentially help in a future classification of all normal Schubert varieties when $p \mid \#\pi_1(G_{\mathrm{der}})$.

It is not clear to us whether tangent spaces of Schubert varieties can be computed in a characteristic-independent way determined by the characteristic 0 description, see Remark 4.4 which comments on the argument in [56, Cor. 4.1].

The key to Theorem 1.1 is to show that the tangent spaces of quasi-minuscule Schubert varieties in twisted affine Grassmannians for absolutely special vertices in characteristic $p > 0$ are big enough, see Proposition 6.1. Thanks to some elementary observations (see Lemmas 5.5 and 5.6) the calculation can be reduced to characteristic 0 where we identify the tangent spaces with those at minimal nilpotent orbits, see Appendix C. This uses the exponential map and representation-theoretic methods. For split groups, this relation to