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CONFIGURATION SPACES OF MANIFOLDS WITH BOUNDARY

par Ricardo CAMPOS, Najib IDRISSI,
Pascal LAMBRECHTS & Thomas WILLWACHER

Abstract. — We study ordered configuration spaces of compact manifolds with boundary. We show that for a large class of such manifolds, the real homotopy type of the configuration spaces only depends on the real homotopy type of the pair consisting of the manifold and its boundary. We moreover describe explicit real models of these configuration spaces using three different approaches. We do this by adapting previous constructions for configuration spaces of closed manifolds which relied on Kontsevich's proof of the formality of the little disks operads. We also prove that our models are compatible with the richer structure of configuration spaces, respectively a module over the Swiss-Cheese operad, a module over the associative algebra of configurations in a collar around the boundary of the manifold, and a module over the little disks operad.

***Résumé.* (Espaces de configuration de variétés à bord)** — Nous étudions les espaces de configuration de variétés compactes à board. Nous démontrons que pour une large classe de telles variétés, le type d'homotopie réelle des espaces de configuration ne dépend que du type d'homotopie réelle de la paire formée par la variété et son bord. Nous décrivons de plus des modèles explicites pour ces espaces de configuration en utilisant trois approches différentes. Pour cela, nous adaptons des constructions précédentes pour les espaces de configuration de variétés compactes sans bord qui reposaient sur la preuve par Kontsevich de la formalité des opérades des petits disques. Nous démontrons de plus que nos modèles sont compatibles avec la structure riche des espaces de configuration, respectivement comme module sur l'opérade Swiss-Cheese, comme module sur l'algèbre associative des configurations dans un collier autour du bord de la variété, et comme module sur l'opérade des petits disques.

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CONTENTS

INTRODUCTION

Configuration spaces of points on manifolds are classical and yet intriguing objects in topology. The ordered configuration space of k points on a space X is given by

$$\mathrm{Conf}_k(X) = \{(x_1, \dots, x_k) \in X^k \mid x_i \neq x_j, \text{ for } i \neq j\}.$$

Despite the apparent simplicity of the definition, understanding their homotopy type, or even their rational homotopy type, has been a long-standing endeavor.

The first results in this direction were obtained by Arnold [**1**] (in 2 dimensions) and Cohen [**13**], who computed the cohomology of the configuration spaces of points in $\mathbb{R}^n$. The real homotopy type of configuration spaces of points on smooth projective varieties were independently computed by Kriz [**33**] and Totaro [**52**].

For closed simply connected manifolds, a way of computing the Betti numbers of these configuration space has been described by Lambrechts and Stanley [**35**]. It has been a long-established conjecture that for such manifolds, the rational homotopy type of the configuration space depends only on the rational homotopy type of the manifold [**17**, Problem 8, p. 518]. For non-simply connected manifolds, this conjecture has a negative answer, as shown in [**40**]. Recently, three of the authors [**9, 29**] proved that the conjecture is true when restricted to *real* homotopy types.

Studying configuration spaces of manifolds with boundary is in some aspects harder, as the their homotopy type should a priori depend on the homotopy types of the manifold, its boundary, and the inclusion between the two. Some results for computing the Betti number of configuration spaces of manifolds with boundary are known [**45**]. However, due to these difficulties, there has not been such a thorough study of the homotopy theory of configuration spaces on manifolds with boundary and in particular, the question of determining whether configuration spaces of compact manifold with boundary is a homotopy invariant remains open. In this work, we prove that, for compact, simply connected manifolds M with simply connected boundary satisfying $\dim M \geq 4$, the real homotopy type of $\mathrm{Conf}_k(M)$ only depends on the real homotopy type of the pair $\partial M \subset M$.

In mathematical physics the study of configuration spaces on manifolds with boundary is also very relevant. For instance, in the BV-BFV formalism [**11, 12**], in order to perturbatively quantize gauge theory in the presence of a boundary, one needs a good understanding of the real homotopy type of configuration spaces on manifolds with boundary. The construction of propagators and the computation of integrals given by Feynman rules admit parallels in these contexts.

The aforementioned results mostly focus on the algebro-topological properties of the configuration spaces on their own. It has long been known, however, that configuration spaces carry rich algebraic structures using "gluing".

More concretely, we consider the operad of little n-disks, initially introduced by Boardman-Vogt [**5**], and which consists of configuration of disjoint n-disks (instead of points) inside the unit n-disk. By considering the centers of each disk, we obtain a homotopy equivalence between this configuration space and the configuration space of points in $\mathbb{R}^n$. However, there is a new algebraic structure on configuration of little disks, namely that of an operad. This operadic structure is given by "composition products," obtained by plugging a configuration of k disks inside one of the disks in a configuration of l disks, to obtain a configuration of $k+l-1$ disks.

Here, for technical reason, we actually use a different model of the little n-disks operad. We consider the Fulton-MacPherson compactification $\mathsf{FM}_n(k)$ of $\mathrm{Conf}_k(\mathbb{R}^n)$ [**2, 21**], obtained by allowing points to become infinitesimally close. The collection $\mathsf{FM}_n = \{\mathsf{FM}_n(k)\}_{k\geq 0}$ can be made into a topological operad, which is equivalent in homotopy to the little n-disks operad.

For a closed parallelized manifold M, there exists a similar compactification FM_M of the configuration space of points on M. This collection FM_M carries the structure of an operadic right FM_n-module, again using insertion of configuration. More generally, for any M, one can define an operad FM_n^M in topological spaces over M, built from fiberwise configuration spaces. Even if M is not parallelizable, the collection FM_M is endowed by the structure of an operadic right module over FM_n^M.

These various operadic structures on configuration spaces have received growing interest over the decades. The configuration spaces of points on manifolds with their operadic module structure have recently seen a surge in interest, due to their central appearance in the Goodwillie-Weiss embedding calculus [**7, 53**] and factorization homology [**3**]. These applications require understanding of the homotopy type of the configuration spaces together with their natural operadic structures.

The first result in this direction was the rational formality of the little disks operads, shown by Tamarkin [**50**] (for $n = 2$, over $\mathbb{Q}$), Kontsevich [**31**] (for all n, over $\mathbb{R}$), with further contributions over the years [**24, 36, 44, 20**].

For closed connected orientable M, the real homotopy types of the configuration spaces FM_M together with the operadic structure have recently computed by three of the authors [**9, 29**], where "workable" combinatorial models were given. This paper is a follow-up to these works, extending the methods and generalizing the results to compact orientable manifolds with boundary.

Summary of results

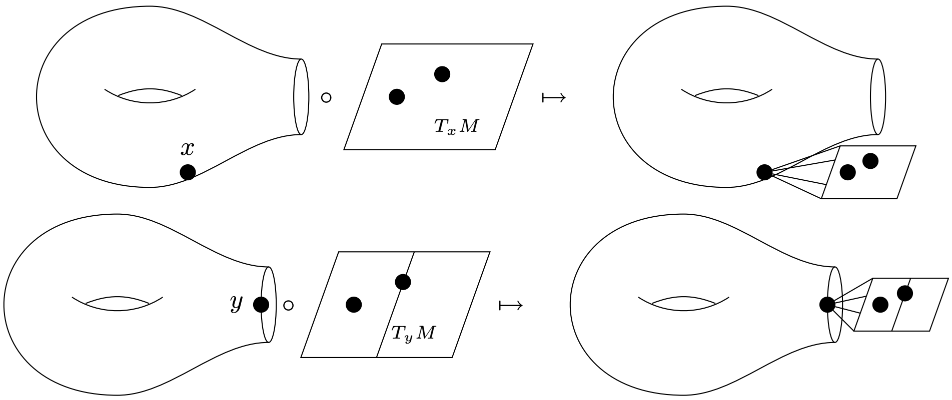
Let M be a compact orientable manifold with non-empty boundary ∂M. We study configuration spaces of r points in the interior and s points on the boundary:

$$\mathrm{Conf}_{r,s}(M) = \{(x_1, \dots, x_r, y_1, \dots, y_s) \times (\partial M)^r \times \mathring{M}^s \mid x_i \neq x_j, y_i \neq y_j \text{ for } i \neq j\}.$$

There are essentially two approaches to defining algebraic structures on those spaces: one that has to do with the action of the Swiss-Cheese operad, and one that has to do with how configuration spaces behave when one glues manifolds along their boundaries. We will describe "graphical" models for both approaches. We will also define "small" models for configurations on the interior of M together with its action of the little disks operad.

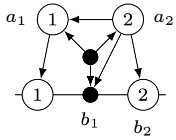
Graphical models: Swiss-Cheese action. — We can compactify $\mathrm{Conf}_{r,s}(M)$ in the spirit of Axelrod-Singer [**2**] to obtain a compact space $\mathsf{SFM}_M(r,s)$, cf. Section 3 below. These compactified spaces come with a natural operadic right action of the fiberwise Swiss-Cheese operad SC_n^M (and the S in SFM stands for "Swiss-Cheese").

More concretely, in the second color, the fiberwise little disks operad FM_n^M acts on SFM_M by insertion of configurations of points "infinitesimally close to" a given point in the configuration. In the first color, we have a similar operation of insertion of configurations of points on the upper half-space, fiberwise over ∂M. The operations are depicted in the following illustration:



Our first main result is the construction of a CDGA model $\mathsf{SGraphs}_{A,A_\partial}$ for the right SFM_n^M-module SFM_M, where (A, A_∂) is a CDGA model for $(M, \partial M)$ (see Section 5.3 for what is precisely expected of (A, A_∂)). The proof mostly follows analogous results and constructions of Kontsevich, a strategy already used in previous works [**9, 29**].

Let us briefly describe this model. Elements of $\mathsf{SGraphs}_{A,A_\partial}(r,s)$ are directed graphs with vertices of 4 kinds: aerial external vertices, numbered from $1, \dots, s$, representing the s points in the interior of M; terrestrial external vertices, numbered $1, \dots, r$, representing the r points on the boundary; and internal "unidentifiable" vertices, either aerial or terrestrial. In addition, aerial vertices may be decorated by elements of A, and terrestrial vertices are decorated by elements of A_∂. Note also that edges may not start at terrestrial vertices. For detailed construction and some more technical conditions we refer the reader to Section 6.



with $a_i \in A, b_j \in A_\partial$.

We can define similarly a graphical model $\mathsf{SGraphs}_n^A$ of the fiberwise Swiss-Cheese operad, and there is a coaction of $\mathsf{SGraphs}_n^A$ on $\mathsf{Graphs}_{A,A_\partial}$ by explicit combinatorial formulas on graphs. Furthermore, all graphical models have a natural dg commutative algebra structure, given by gluing diagrams at external vertices.

Our first main result is then that these graphical objects are indeed models for the topological configuration spaces, in the sense that they are quasi-isomorphic to the CDGAs of differential forms on those spaces:

***Theorem A* (See Theorem 6.23). —** *Let M be an oriented compact manifold with boundary $\partial M \neq \emptyset$. Then there is zigzag of quasi-isomorphisms relating the pairs*

$$(\mathsf{SGraphs}_{A,A_\partial}, \mathsf{SGraphs}_n^A, \mathsf{Graphs}_n^A) \simeq (\Omega_{\mathrm{PA}}(\mathsf{SFM}_M), \Omega_{\mathrm{PA}}(\mathsf{SFM}_n^M), \Omega_{\mathrm{PA}}(\mathsf{FM}_n^M))$$

compatible with all structures, i.e., with the dg commutative algebra structure and the operadic action of the second member of the pairs on the first.

We note that the graded object $\mathsf{SGraphs}_{A,A_\partial}$ depends on M only through the homotopy type of M, while that is certainly not true for the real homotopy type of the configuration spaces. The dependence on M in Graphs_M comes from the differential. More concretely, the differential, and hence all dependence on M, is neatly encoded by a Maurer-Cartan element Z_M in a certain graph complex. Physically, this Maurer-Cartan element corresponds to the partition function in the underlying topological field theory, taking values in the complex of vacuum Feynman diagrams. In this paper we will hence call this special MC element Z_M that governs the real homotopy type of our configuration spaces the "partition function," although we will not discuss any connections to physics.

The partition function can actually be evaluated under good conditions. In particular, if M and ∂M are simply connected, and $\dim(M) \geq 5$, then the partition function only depends on the real homotopy type of M:

***Corollary B* (Corollary 6.29). —** *If M and ∂M are simply connected, and* $\dim(M) \geq 5$ *then the real homotopy type of SFM_M (as space, and as right module under the fiberwise Swiss-Cheese operad) only depends on the real homotopy type of the map $\partial M \to M$.*

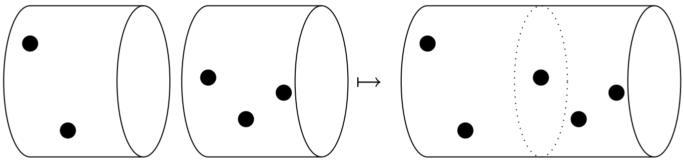
More concretely, the real homotopy type is precisely encoded in the tree piece of the partition function. The above result hence means that there are no "quantum corrections".

Note that for $\dim M \leq 4$, using the (generalized) Poincaré conjecture (see Freedman **[18]** for $n = 4$ and Perelman **[43, 42]** for $n = 3$), if M and its boundary are simply connected then M is homeomorphic to D^3 or D^4 (or, depending on convention, D^0 if $\varnothing$ is considered simply connected). Therefore, in dimension ≤ 4, if $(M, \partial M)$ and $(M', \partial M')$ are homotopy equivalent, then they are homeomorphic. Since homeomorphic spaces have homeomorphic configuration spaces, Corollary B thus holds vacuously for the real homotopy type of the *spaces* SFM_M, as there is at most one possible manifold in each dimension to consider. However, the right action of the fiberwise Swiss-Cheese operad depends on tangent spaces, and thus the diffeomorphism types of the manifold. In dimension ≤ 3, M must also be diffeomorphic to D^3, so Corollary B holds in full. The existence of exotic $\mathbb{R}^4$ prevents us from concluding in dimension 4 and we do not know if there exists a counterexample to real homotopy invariance in dimension 4.

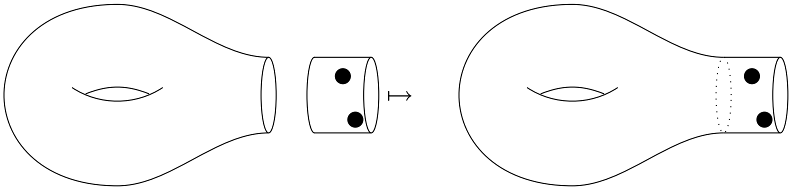
Graphical models: gluing at the boundary. — There is an alternative viewpoint on the configuration spaces of points on manifolds with boundary, that both gives rise to simpler models, and to algebraic structure which is not easily extracted from (although contained in) the Swiss-Cheese action above. Let us now only consider configuration spaces

$$\mathrm{Conf}_r(M) \coloneqq \mathrm{Conf}_{0,r}(M)$$

of points in the interior, with no points on the boundary. Also consider the configuration space of points on $\partial M \times I$, i.e., $\mathrm{Conf}_{r'}(\partial M \times I)$, where $I = (0, +\infty)$. The collection of the latter spaces (for various r') naturally forms an algebra object (more precisely, an E_1-algebra object), the product being the gluing of the intervals



This E_1-algebra naturally acts on $\mathrm{Conf}_M(r)$ by gluing at the boundary:



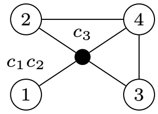
Understanding these gluing operations is of high importance, because they allow the configuration space of points on a glued manifold $X = M \sqcup_{\partial M} N$ to be expressed through the configuration spaces of the pieces, as a "derived tensor product"

$$\mathrm{Conf}(X) \simeq \mathrm{Conf}(M) \otimes^{\mathbb{L}}_{\mathrm{Conf}(\partial M \times I)} \mathrm{Conf}(N).$$

We refer the reader to [**3**] for more explanations.

The algebraic operations above (of algebra and module, by gluing at the boundary) are encoded in our models in the Swiss-Cheese action, but not in an accessible form. We may however describe simpler models in this case that capture those gluing operations more nicely. To this end it is also suitable to define slightly different compactifications mFM_M and $\mathsf{aFM}_{\partial M}$ (where "a" stands for "algebra" and "m" for module, see Section 3.7). The algebra and module structures described above are defined on the nose for these models of the configuration spaces.

In this setting we can construct significantly simpler combinatorial models for our configuration spaces $\mathsf{mGraphs}_A$ and $\mathsf{aGraphs}_{A_\partial}$. Concretely, elements of $\mathsf{mGraphs}_A(r)$ are directed graphs with only two types of vertices, external vertices numbered $1 \dots, r$, and internal vertices. All vertices are decorated by an element on A.



with $c_j \in A$.

The construction of $\mathsf{aGraphs}_{A_\partial}$ is similar.

All the algebraic operations may then be encoded combinatorially on these diagrams. Our second main result is then:

***Theorem C* (See Section 7).** — *For M a compact oriented manifold with boundary ∂M, we have a zig-zag of quasi-isomorphisms*

$$(\mathsf{mGraphs}_A, \mathsf{aGraphs}_{A_\partial}, \mathsf{Graphs}_n^A) \simeq (\Omega_{\mathrm{PA}}(\mathsf{mFM}_M), \Omega_{\mathrm{PA}}(\mathsf{aFM}_{\partial M}), \Omega_{\mathrm{PA}}(\mathsf{FM}_n^M))$$

respecting all algebraic structures, i.e., the CDGA structures, the operadic right actions, and the E_1-algebra and module structure obtained by gluing at the boundary.

Again, the objects here depend on a certain Maurer-Cartan element in a graph complex called the partition function. This partition function can can be evaluated under good conditions, just like for SFM_M. The conditions here are weaker than those of Corollary B (see page 85), which can heuristically be explained because the graph complex for SFM_M depends on a model for $\mathrm{Conf}(\partial M)$, which requires $\dim \partial M \geq 4 \iff \dim M \geq 5$.

Corollary D. — *If M is simply connected, and $\dim(M) \geq 4$ then the real homotopy type of mFM_M (as space, and as right module over aFM_N) only depends on the real homotopy type of the map $\partial M \to M$.*

In Section 8.4, we connect the two models $\mathsf{SGraphs}_{A,A_\partial}(0,-)$ (all the external vertices are in the interior) and $\mathsf{mGraphs}_A$ as comodules over Graphs_n^A. The only possible manifold in dimension ≤ 3 is D^3 by the Poincaré conjecture, so the corollary holds vacuously in low dimensions.

Small models and coaction of the cohomology. — Under some hypotheses about the connectivity and the dimension of M, we will also find some "small" models for $\mathrm{Conf}_k(M)$, inspired by the Lambrechts-Stanley models for configuration spaces of closed manifolds (see [**29**] and [**9**, Appendix A]).

Suppose both M and ∂M are simply connected and that $\dim M \geq 7$, so that the pair $(M, \partial M)$ admits a Poincaré-Lefschetz duality model, a notion we define in Section 2, and let P be the resulting model of M. We may then use the same construction as in [**29**] to get a CDGA $\mathsf{G}_P(k)$. If $\partial M = \varnothing$, then P is a Poincaré duality model of M and we recover the Lambrechts-Stanley model of $\mathrm{Conf}_k(M)$. We show in Theorem 8.9 that there is an isomorphism of graded vector spaces between $H^*(\mathsf{G}_P(k))$ and $H^*(\mathrm{Conf}_k(M))$ over $\mathbb{Q}$, which generalizes the result of [**35**].

However, $\mathsf{G}_P(k)$ is not a CDGA model of $\mathrm{Conf}_k(M)$ in general. Instead we consider a "perturbed" version $\tilde{\mathsf{G}}_P(k)$, which is isomorphic to $\mathsf{G}_P(k)$ as a dg-module but not as an algebra. We show that $\tilde{\mathsf{G}}_P(k)$ is a CDGA model of $\mathrm{Conf}_k(M)$. Moreover, we prove that $\tilde{\mathsf{G}}_P$ is a right Hopf $\mathsf{e}_n^\vee$-comodule if $\partial M \neq \varnothing$, and if M is framed then we prove that the quasi-isomorphism $\tilde{\mathsf{G}}_P \simeq \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\varnothing, -))$ is compatible with the comodule structures (over $\mathsf{e}_n^\vee$ and $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$, respectively).

***Theorem E* (See Theorems 8.19–8.20). —** *Let M be a smooth, simply connected connected compact n-manifold with simply connected boundary of dimension at least 5. Assume that either M admits a surjective pretty model, or that $n \geq 7$ so that M admits a Poincaré-Lefschetz duality model. Let P be the model built either out of the surjective pretty model or the Poincaré-Lefschetz duality model.*

Then for all $k \geq 0$, the CDGA $\tilde{\mathsf{G}}_P(k)$ is weakly equivalent to $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(0, k))$, and the equivalence is compatible with the action of the symmetric group Σ_k; in particular, it is a model of $\mathrm{Conf}_k(M)$. Moreover, if M is framed, then the right Hopf comodule $(\tilde{\mathsf{G}}_P, \mathsf{e}_n^\vee)$ is weakly equivalent to $(\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(0, -)), \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n))$.

The same result holds with $P = H^*(M)$ for simply connected manifolds with simply connected boundary satisfying $\dim M \in \{4, 5, 6\}$.

The advantage of this small model is that it can be used to do some computations, e.g., of factorization homology (see [**29**, Section 5]) or embedding calculus. Note that despite the notation, $\tilde{\mathsf{G}}_P$ depends not just on the model P of M but on the full Poincaré-Lefschetz duality model of the pair $(M, \partial M)$.

***Remark F.* —** *All of our models are compatible with the symmetric group actions. Therefore, we obtain models of the unordered configuration spaces $B_k(M) = \mathrm{Conf}_k(M)/\Sigma_k$ by considering the sub-CDGA of elements invariant under the symmetric group action. Note however that the unordered configuration spaces are not simply connected even if M is (if $\dim M \geq 3$ and M is simply connected then $\pi_1(B_k(M)) = \Sigma_k$) so this may give less information than expected. This is still sufficient to compute the cohomology, for example.*

Outline

		Closed mfd	Swiss-Cheese	E_1-algebra	E_1-module
Compactif.	local	FM_n §3.2	SFM_n §3.5	n/a	n/a
	fibered	FM_n^M §3.3	SFM_n^M §3.5	n/a	n/a
	global	FM_M §3.4	SFM_M §3.6	$\mathsf{aFM}_{\partial M}$	& mFM_M §3.7
Propagator	local	[31]	[57]	n/a	n/a
	fibered	[8]	§6.1	n/a	n/a
	global	§4.1	§4.3	§4.4	§4.5
Model	local	Graphs_n §5.1	$\mathsf{SGraphs}_n$ §5.1	n/a	n/a
	fibered	Graphs_n^A §5.5	$\mathsf{SGraphs}_n^A$ §6.1	n/a	n/a
	global	Graphs_A §5.3	$\mathsf{SGraphs}_{A,A_\partial}$ §6	$\mathsf{aGraphs}_{A_\partial}$	& $\mathsf{mGraphs}_A$ §7.1
MC elements	local	$\mu \in \mathrm{GC}_n^\vee$ (5.2)	$c \in \mathrm{SGC}_n^\vee$ (5.7)	n/a	n/a
	fibered	$z \in A \mathbin{\hat{\otimes}} \mathrm{GC}_n^\vee$ §5.6	$z^\partial \in A_\partial \mathbin{\hat{\otimes}} \mathrm{SGC}_n^\vee$ §6.1	n/a	n/a
	global	$Z \in \mathrm{GC}_A^\vee$	$Z \in \mathrm{SGC}_{A,A_\partial}^\vee$ §6.5	$w \in \mathrm{aGC}_{A_\partial}^\vee$	$W \in \mathrm{mGC}_A^\vee$ §7.2

Section 1 : We recall some background on cooperads and comodules over them, operads over spaces, the cohomology of compact manifolds with boundary, and pretty models.

Section 2 : We define Poincaré-Lefschetz duality models, a generalization of surjective pretty models, and we prove that any simply connected manifold with simply connected boundary of dimension at least 7 admits a Poincaré-Lefschetz duality model.

Section 3 : We recall and define various compactifications for configuration spaces of Euclidean (half-)spaces and manifolds with and without boundary, inspired by the Axelrod-Singer-Fulton-MacPherson compactifications.

Section 4 : We explain how to construct the "propagators" which will be used to define integrals on these compactified configuration spaces, using the usual Feynman rules.

Section 5 : We recall the construction of models for configuration spaces of closed manifolds [9, 29] that we will generalized for compact manifolds with boundary. We also explain in what sense the graphical models we build are "functorial," which will be used in the rest of the paper.

Section 6 : We build our first graphical model $\mathsf{SGraphs}_{A,A_\partial}$, and we prove that it is a model of $\mathrm{Conf}_{\bullet,\bullet}(M)$ as an operadic module over the Swiss-Cheese operad.

Section 7 : We build our second graphical model, $\mathsf{mGraphs}_A$, and we prove that it is a model of $\mathrm{Conf}_{0,\bullet}(M)$ as a module over the E_1-algebra $\mathrm{Conf}_{\partial M \times \mathbb{R}_{>0}}$.

Section 8 : We build a first small dg-module $\mathsf{G}_P(k)$, and we prove that under some hypotheses, it computes the Betti numbers of $\mathrm{Conf}_k(M)$. We then prove that a "perturbed" variant $\tilde{\mathsf{G}}_P(k)$ is a CDGA model for $\mathrm{Conf}_k(M)$ as a module over the cohomology $\mathsf{e}_n^\vee$ of the little n-disks operad. We also make precise the connection between $\mathsf{SGraphs}_{A,A_\partial}$ and $\mathsf{mGraphs}_A$.

Appendix : We compute the cohomology of several graph complexes that appear throughout the paper.

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