

THE SPACE OF GENERALIZED FORMAL POWER SERIES SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION

by

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Abstract. — We prove that the set of truncations of generalized power series solutions of an ordinary differential equations is contained in a semi-algebraic set of dimension bounded by twice the order of the differential equation.

Résumé (L'espace des séries formelles généralisées qui sont solution d'une équation différentielle ordinaire)

Nous montrons que l'ensemble des troncations de séries généralisées qui sont solutions d'une équation différentielle ordinaire est contenu dans un ensemble semi-algébrique dont la dimension est bornée par le double de l'ordre de l'équation différentielle.

1. Introduction

Consider a polynomial differential equation $F(\partial_0(y), \dots, \partial_n(y)) = 0$, where $F(y_0, \dots, y_n)$ is a polynomial in the variables y_0, \dots, y_n with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$ (polynomials with real exponents). We are interested in series solutions of $(F = 0)$ of the form $\sum_{i=1}^{\infty} c_i x^{\mu_i}$, where $c_i \in \mathbb{C}$ and $\mu_i \in \mathbb{R}$ with $\mu_1 < \mu_2 < \dots$ (so called *generalized power series*). D.Y. Grigor'ev and M. Singer describe in [5] a parametric version of the Newton polygon process applied to F , which for each integer k , gives rise to a semi-algebraic subset $\text{NIC}_k^*(F) \subseteq \mathbb{R}^{3k}$ so that the space of truncations of length k of generalized power series solution of $(F = 0)$ is included in $\text{NIC}_k^*(F)$. The main contribution of this paper is to prove that the dimension of this semi-algebraic set is bounded by $2n$. More precisely, its *adapted dimension* (see subsection 3.2) is bounded by n . The adapted dimension is a proper measure of the *number of free parameters* (real or complex, coefficient or exponent) which have been introduced

2010 Mathematics Subject Classification. — 37F75.

Key words and phrases. — ODE, generalized series.

The authors wish to thank the referee for many helpful suggestions.

along the Newton polygon process in a parametric family of power series solution of a differential equation.

Briot and Bouquet [1] in 1856 use the Newton polygon for studying first order and first degree ordinary differential equations and Fine [4] in 1889 gives a description of the method for ordinary differential equation of arbitrary order. In section 2 we present a brief introduction to its classical version. In section 4 we introduce the notion of parametric Newton polygon: specifically, we define it and give some technical results about *parametric polynomials* which will be used in the proof of the main theorem.

In section 3 we state the main theorem and give a straightforward proof for the case $k = 1$. The general case is dealt with in section 5.

2. Newton polygon of an ODE

A *well-ordered* series with complex coefficients and real exponents is a series $\phi(x) = \sum_{\alpha \in S} c_\alpha x^\alpha$, where $c_\alpha \in \mathbb{C}$, and S is a well ordered subset of \mathbb{R} . If there exist a finitely generated semi-group Γ of $\mathbb{R}_{\geq 0}$ and $\gamma \in \mathbb{R}$, such that, $S \subseteq \gamma + \Gamma$, then we say that $\phi(x)$ is a *grid-based* series (this terminology comes from [6].) Let $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ be the sets of well-ordered series and of grid-based series, respectively. We denote $\mathbb{C}[x^{\mathbb{R}}]$ the subring of series in $\mathbb{C}((x))^g$ with finite support (*polynomials*, so to speak). It is well-know (see [7], for example), that both $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ are actually fields. Both are differential rings with the usual inner operations and the differential operator $\partial = x \frac{d}{dx}$:

$$\partial \left(\sum c_\alpha x^\alpha \right) = \sum \alpha c_\alpha x^\alpha.$$

Denote by ∂_0 the identity operator and for positive integer i , $\partial_i = \partial \circ \partial_{i-1}$.

Let $F(y_0, \dots, y_n)$ be a polynomial in the variables y_0, \dots, y_n with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$. The differential equation

$$F(\partial_0(y), \partial_1(y), \dots, \partial_n(y)) = 0$$

will be denoted by $F(y) = 0$. Notice that any polynomial ordinary differential equation can be rewritten in this form.

We are interested in solutions of $F(y) = 0$ in the field $\mathbb{C}((x))^w$. By virtue of [2, 5, 6], all of them are actually in $\mathbb{C}((x))^g$.

Write F in a uniquely, using the standard multiindex notation $y^\rho = y_0^{\rho_0} \cdots y_n^{\rho_n}$ (where $\rho = (\rho_0, \dots, \rho_n)$) as

$$F = \sum_{\alpha, \rho} A_{\alpha, \rho} x^\alpha y^\rho, \text{ with } A_{\alpha, \rho} \in \mathbb{C},$$

where α and ρ run over finite subsets of \mathbb{R} and \mathbb{N}^{n+1} respectively. The *cloud of points* of F is the set

$$\mathcal{P}(F) = \{(\alpha, |\rho|) : A_{\alpha, \rho} \neq 0\},$$

where $|\rho| = \rho_0 + \dots + \rho_n$. The Newton polygon $\mathcal{N}(F)$ of F is the convex hull of

$$\bigcup_{P \in \mathcal{P}(F)} (P + \{(a, 0) \mid a \geq 0\}) .$$

Notice that $\mathcal{N}(F)$ has a finite number of vertices, all of whose ordinates are non-negative integers.

Given a line $L \subseteq \mathbb{R}^2$ with slope $-1/\mu$, we say that μ is the *inclination* of L . Let $\mu \in \mathbb{R}$, we denote $L(F; \mu)$ the supporting line of $\mathcal{N}(F)$ with inclination μ (i.e. the only line L with inclination μ such that $\mathcal{N}(F)$ is contained in the right closed half-plane defined by L and $L \cap \mathcal{N}(F) \neq \emptyset$). More precisely, $L(F; \mu)$ is the set of points (a, b) in \mathbb{R}^2 such that $a + \mu b = \nu(F; \mu)$, where $\nu(F; \mu) = \min\{\alpha + \mu|\rho|; A_{\alpha, \rho} \neq 0\}$.

For any $\mu \in \mathbb{R}$, define the polynomial

$$(1) \quad \Phi_{(F; \mu)}(\mathbf{c}) = \sum_{(\alpha, |\rho|) \in L(F; \mu)} A_{\alpha, \rho} \mu^{w(\rho)} \mathbf{c}^{|\rho|} \in \mathbb{C}[\mathbf{c}],$$

where $w(\rho) = \rho_1 + 2\rho_2 + \dots + n\rho_n$. The *Newton polygon data* of F will be the set of vertices v_0, \dots, v_t (ordered with decreasing ordinate), the sides $[v_i, v_{i+1}]$, $0 \leq i < t$, the *indicial polynomials* associated to each vertex v :

$$(2) \quad \Psi_{(F; v)}(\mathbf{m}) = \sum_{(\alpha, |\rho|) = v} A_{\alpha, \rho} \mathbf{m}^{w(\rho)} \in \mathbb{C}[\mathbf{m}].$$

and the *characteristic polynomials* associated to each side $[v_i, v_{i+1}]$:

$$\Phi_{(F; [v_i, v_{i+1}])}(\mathbf{c}) = \Phi_{(F; \mu_{[v_i, v_{i+1}]})}(\mathbf{c}),$$

where $\mu_{[v_i, v_{i+1}]}$ is the inclination of side $[v_i, v_{i+1}]$.

2.1. Necessary Initial Conditions. — Given a well-ordered formal power series $y(x) = \sum_{\alpha \in S} c_\alpha x^\alpha$, its *order*, $\text{ord}(y(x))$, is infinity if $y(x) = 0$ and $\min\{\alpha \in S \mid c_\alpha \neq 0\}$ otherwise.

Lemma 1. — *Let $y(x) = cx^\mu + \sum_{\alpha > \mu} c_\alpha x^\alpha \in \mathbb{C}((x))^w$ be a solution of the differential equation $F(y) = 0$. Then*

$$\Phi_{(F; \mu)}(c) = 0.$$

where c may be zero. In particular, if $y(x) = 0$ is a solution of $F(y) = 0$ then $\Phi_{(F; \mu)}(0) = 0$ for all μ .

Proof. — Developing F

$$\begin{aligned} F(cx^\mu + \dots) &= \\ \sum_{\alpha, \rho} A_{\alpha, \rho} x^\alpha (cx^\mu + \dots)^{\rho_0} (\mu cx^\mu + \dots)^{\rho_1} \dots (\mu^n cx^\mu + \dots)^{\rho_n} &= \\ \sum_{\alpha, \rho} \{A_{\alpha, \rho} c^{|\rho|} \mu^{w(\rho)} x^{\alpha + \mu|\rho|} + \dots\} &= \\ \left\{ \sum_{\alpha + \mu|\rho| = \nu(F; \mu)} A_{\alpha, \rho} c^{|\rho|} \mu^{w(\rho)} \right\} x^{\nu(F; \mu)} + \dots, \end{aligned}$$

where dots \dots stand for monomials of order greater than the exponent of x in the preceding term. The lemma follows from the fact that $\alpha + \mu|\rho| = \nu(F; \mu)$ if and only if $(\alpha, |\rho|) \in L(F; \mu)$. \square

Notation 1. — Let $\varphi \in \mathbb{C}((x))^g$ and $F(y_0, \dots, y_n) \in \mathbb{C}((x))^g[y_0, \dots, y_n]$, denote

$$F(\varphi + y) = F(\varphi + y_0, \partial(\varphi) + y_1, \dots, \partial_n(\varphi) + y_n) \in \mathbb{C}((x))^g[y_0, \dots, y_n].$$

Definition 1. — Given $F(y_0, \dots, y_n)$ and a positive integer k , define the set of necessary k -initials conditions, $\text{NIC}_k(F)$, to be the subset of $(\mathbb{R} \times \mathbb{C})^k$ of the points $(\mu_1, c_1, \dots, \mu_k, c_k) \in (\mathbb{R} \times \mathbb{C})^k$ such that

$$\mu_1 < \dots < \mu_k, \text{ and}$$

$$\Phi_{(F_1; \mu_1)}(c_1) = 0, \dots, \Phi_{(F_k; \mu_k)}(c_k) = 0,$$

where $F_1(y) = F(y)$ and $F_{i+1}(y) = F_i(c_i x^{\mu_i} + y)$, for $1 \leq i < k$.

Define the $\text{NIC}_k^*(F) = \text{NIC}_k(F) \cap (\mathbb{R} \times \mathbb{C}^*)^k$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Corollary 1. — If $y(x) = \sum_{i=1}^k c_i x^{\mu_i} + \sum_{\mu_k < \alpha} c_\alpha x^\alpha$ is a solution of $F(y) = 0$ with $\mu_1 < \dots < \mu_k$, then

$$(\mu_1, c_1, \dots, \mu_k, c_k) \in \text{NIC}_k(F).$$

Corollary 2. — Let v_0, \dots, v_t be the vertices of $\mathcal{N}(F)$, ordered by decreasing ordinate. Let $\mu_i, 1 \leq i \leq t$ be the inclination of the side $[v_{i-1}, v_i]$. Set $\mu_0 = -\infty$ and $\mu_{t+1} = +\infty$. The subset $\text{NIC}_1(F) \subseteq (\mathbb{R} \times \mathbb{C})$ is semi-algebraic. Moreover, $\text{NIC}_1^*(F)$ is the finite union of the semi-algebraic sets corresponding to the sides of the Newton polygon of F :

$$\{(\mu, c) \in \mathbb{R} \times \mathbb{C}^*; \mu = \mu_i, \text{ and } \Phi_{(F; \mu_i)}(c) = 0\}, \quad 1 \leq i \leq t,$$

and the semi-algebraic sets corresponding to the vertices:

$$\{(\mu, c) \in \mathbb{R} \times \mathbb{C}^*; \mu_i < \mu < \mu_{i+1}, \text{ and } \Psi_{(F; v_i)}(\mu) = 0\}, \quad 0 \leq i \leq t.$$

Proof. — Let $\mu \in \mathbb{R}$, $\mu_i < \mu < \mu_{i+1}$, for some $0 \leq i \leq t$. As $L(F; \mu) \cap \mathcal{N}(F) = v_i$ and $\Phi_{(F; \mu)}(c) = c^h \Psi_{(F; v_i)}(\mu)$, (where h is the ordinate of v_i) then, for $c \neq 0$ and $\mu_i < \mu < \mu_{i+1}$, one has $\Phi_{(F; \mu)}(c) = 0$ if and only if $\Psi_{(F; v_i)}(\mu) = 0$, and we are done. \square

Let $\mu \in \mathbb{R}$ be a real number and fix a point $(a, h) \in \mathbb{R} \times \mathbb{N}$.

Definition 2. — We say that (a, h) belongs to the red part with respect to μ of the Newton polygon of $F(y)$ if $h \geq 1$ and either (a, h) is the vertex of $\mathcal{N}(F)$ with minimum ordinate or it belongs to a side of $\mathcal{N}(F)$ with inclination greater than μ .

Notice that if the red part with respect to μ of $\mathcal{N}(F)$ is empty, then there are no generalized power series solution of $(F = 0)$ of order greater than μ : the vertex (a, h) with minimum ordinate has $h = 0$ and all the sides of $\mathcal{N}(F)$ have inclination less than or equal to μ , hence for $\gamma > \mu$, the polynomial $\Phi_{(F;\mu)}(c)$ is a non-zero constant and by Corollary 2 the set $\text{NIC}_1^*(F)$ is empty. The reciprocal is not true as Example 1 (page 65) shows.

Lemma 2. — Let $(\mu_1, c_1, \dots, \mu_k, c_k) \in \text{NIC}_k^*(F)$, $\varphi = \sum_{j=1}^k c_j x^{\mu_j}$ and $F_{k+1}(y) = F(\varphi + y)$. The red part of $\mathcal{N}(F_{k+1}(y))$ with respect to μ_k nonempty.

Proof. — Let $(\mu, c) \in \text{NIC}_1^*(F)$ and consider $G = F(cx^\mu + y)$. The red part of the Newton polygon of G with respect to μ is not empty. To see this, let v_0, \dots, v_t be the vertices of $\mathcal{N}(F)$ ordered by decreasing ordinate and let v_k be the vertex with highest ordinate in $L(F; \mu) \cap \mathcal{N}(F)$. The ordinate of this v_k is greater than zero because otherwise $\Phi_{(F;\mu)}(c)$ would be a nonzero constant, in contradiction with the fact that $\Phi_{(F;\mu)}(c) = 0$.

Returning to the main argument, given a monomial $M = x^\alpha y_0^{\rho_0} \dots y_n^{\rho_n}$, one may write

$$(3) \quad M(cx^\mu + y) = x^\alpha \prod_{i=0}^n (c\mu^i x^\mu + y_i)^{\rho_i} = M + R,$$

where the points corresponding to the monomials of R have ordinate less than $|\rho|$ and belong to the line with inclination μ passing through $(\alpha, |\rho|)$. If w is the intersection of $L(F; \mu)$ with the axis of abscissas, then the cloud of points $\mathcal{P}(G)$ of G is contained in the positive convex hull of $\{v_0, \dots, v_k, w\}$. The coefficient of G corresponding to w is precisely $\Phi_{(F;\mu)}(c) = 0$, hence $w \notin \mathcal{P}(G)$. Moreover, $\{v_0, \dots, v_k\} \subseteq \mathcal{P}(G)$, because of (3). Therefore v_0, \dots, v_k are vertices of $\mathcal{N}(G)$. Hence either v_k is the vertex of $\mathcal{N}(G)$ with minimum ordinate or there exists a side of $\mathcal{N}(G)$ with inclination greater than μ and we are done. \square

Example 1 (See Figure 1). — Let $F = x^{-1} y_0^6 y_1 + y_0^2 y_1 + x y_0^2 - 3 x y_0 y_1 - x^2 y_0 + 2 x^2 y_1 + x^5$. The point $(1, 1) \in \text{NIC}^*(F)$. Let $G = F(x + y)$. The red part of $\mathcal{N}(G)$ with respect to $\mu = 1$ is vertex v_2' and point p . In this example, there are no solutions of $(G = 0)$ of order greater than 1.