Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

INFINITESIMAL MORITA HOMOMORPHISMS AND THE TREE-LEVEL OF THE LMO INVARIANT

Gwénaël Massuyeau

Tome 140
Fascicule 1

2012

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 101-161

Bull. Soc. math. France
140 (1), 2012, p. 101–161

INFINITESIMAL MORITA HOMOMORPHISMS AND THE TREE-LEVEL OF THE LMO INVARIANT

BY GWÉNAËL MASSUYEAU

ABSTRACT. — Let Σ be a compact connected oriented surface with one boundary component, and let π be the fundamental group of Σ. The Johnson filtration is a decreasing sequence of subgroups of the Torelli group of Σ, whose k-th term consists of the self-homeomorphisms of Σ that act trivially at the level of the k-th nilpotent quotient of π. Morita defined a homomorphism from the k-th term of the Johnson filtration to the third homology group of the k-th nilpotent quotient of π.

In this paper, we replace groups by their Malcev Lie algebras and we study the "infinitesimal" version of the k-th Morita homomorphism, which is shown to correspond to the original version by a canonical isomorphism. We provide a diagrammatic description of the k-th infinitesimal Morita homomorphism and, given an expansion of the free group π that is "symplectic" in some sense, we show how to compute it from Kawazumi's "total Johnson map".

Besides, we give a topological interpretation of the full tree-reduction of the LMO homomorphism, which is a diagrammatic representation of the Torelli group derived from the Le–Murakami–Ohtsuki invariant of 3-manifolds. More precisely, a symplectic expansion of π is constructed from the LMO invariant, and it is shown that the tree-level of the LMO homomorphism is equivalent to the total Johnson map induced by this specific expansion. It follows that the k-th infinitesimal Morita homomorphism coincides with the degree $[k, 2k[$ part of the tree-reduction of the LMO homomorphism. Our results also apply to the monoid of homology cylinders over Σ.

Texte reçu le 27 décembre 2010, accepté le 29 juin 2011.

GWÉNAËL MASSUYEAU, IRMA, Université de Strasbourg & CNRS, 7 rue René Descartes, 67084 Strasbourg, France • *E-mail :* massuyeau@math.unistra.fr

2010 Mathematics Subject Classification. — 57M27, 57R19, 57R50, 20F28, 20F38, 20F40.

Key words and phrases. — Torelli group, Johnson homomorphisms, Morita homomorphisms, Magnus expansion, Malcev Lie algebra, homology cylinder, finite-type invariant, LMO invariant.

Partially supported by the French ANR research project ANR-08-JCJC-0114-01.

Résumé (*Homomorphismes de Morita infinitésimaux et réduction arborée de l'invariant LMO*)

Soit Σ une surface compacte orientée avec une composante de bord, et soit π le groupe fondamental de Σ. La filtration de Johnson est une suite décroissante de sous-groupes du groupe de Torelli de Σ, dont le k-ième terme est constitué de tous les homéomorphismes de Σ agissant trivialement au niveau du k-ième quotient nilpotent de π. Morita a défini un homomorphisme du k-ième terme de la filtration de Johnson vers le troisième groupe d'homologie du k-ième quotient nilpotent de π.

Dans cet article, nous remplaçons les groupes par leurs algèbres de Lie de Malcev et nous étudions une version « infinitésimale » du k-ième homomorphisme de Morita, que nous montrons être équivalente à la version originale par un isomorphisme canonique. Nous apportons une description diagrammatique du k-ième homomorphisme de Morita infinitésimal et, étant donné un développement du groupe libre π qui est « symplectique » en un sens, nous montrons comment cet homomorphisme peut être calculé à partir de l'« application de Johnson totale » introduite par Kawazumi.

En outre, nous donnons une interprétation topologique de toute la réduction arborée de l'homomorphisme LMO, qui est une représentation diagrammatique du groupe de Torelli obtenue de l'invariant de Le–Murakami–Ohtsuki des variétés de dimension trois. Plus précisément, un développement symplectique de π est construit à partir de l'invariant LMO, et nous montrons que la réduction arborée de l'homomorphisme LMO est équivalente à l'application de Johnson totale correspondant à ce développement. Il en découle que le k-ième homomorphisme de Morita coïncide avec la troncation en degré $[k, 2k[$ de la réduction arborée de l'homomorphisme LMO. Nos résultats s'appliquent aussi au monoïde des cylindres d'homologie sur Σ.

Introduction

Nilpotent homotopy types of 3-manifolds have been introduced by Turaev [42]. They are defined by elementary tools from algebraic topology as follows. We fix an integer $k \geq 1$ and an abstract group G of nilpotency class k, which means that commutators of length $(k+1)$ are trivial in G. Let M be a closed connected oriented 3-manifold, whose k-th nilpotent quotient of the fundamental group is parametrized by the group G:

$$\psi : G \xrightarrow{\simeq} \pi_1(M)/\Gamma_{k+1}\pi_1(M).$$

Then, the *k-th nilpotent homotopy type* of the pair (M, ψ) is the homology class

$$\mu_k(M, \psi) := f^{\psi}_*([M]) \in H_3(G; \mathbb{Z})$$

where $f^{\psi} : M \to \mathrm{K}(G, 1)$ induces the composition

$$\pi_1(M) \to \pi_1(M)/\Gamma_{k+1}\pi_1(M) \stackrel{\psi^{-1}}{\to} G$$

at the level of fundamental groups. For example, for $k = 1$, we are considering the *abelian homotopy type* of 3-manifolds which, by the work of Cochran, Gerges and Orr [8], is very well understood: $\mu_1(M,\psi)$ determines the cohomology ring of M together with its linking pairing, and vice versa.

As suggested to the author by Turaev, one way to study the invariant μ_k for higher k is to study its behaviour under surgery. This method particularly applies if one wishes to understand nilpotent homotopy types from the point of view of finite-type invariants, which was our initial motivation. Note that, to compare the k-th nilpotent homotopy type of a manifold after surgery with that of the manifold before surgery, we can only admit surgeries that preserve the k-th nilpotent quotient of the fundamental group (up to isomorphism). The following type of surgery is admissible in that sense. We consider a compact connected oriented surface $S \subset M$ with one boundary component, and a homeomorphism $s : S \to S$ whose restriction to ∂S is the identity and which acts trivially at the level of $\pi_1(S)/\Gamma_{k+1}\pi_1(S)$. Then, we can "twist" M along S by s to obtain the new manifold

$$M_S := (M \setminus \mathrm{int}(S \times [-1,1])) \cup_{(s\times 1)\cup(\mathrm{Id}\times(-1))} S \times [-1,1].$$

The Seifert–Van Kampen theorem shows the existence of a canonical isomorphism

$$\pi_1(M)/\Gamma_{k+1}\pi_1(M) \xrightarrow{\simeq} \pi_1(M_S)/\Gamma_{k+1}\pi_1(M_S)$$

which is defined by the following commutative diagram:

$$\begin{array}{ccc} & \dfrac{\pi_1(M\setminus\mathrm{int}(S\times[-1,1]))}{\Gamma_{k+1}\pi_1(M\setminus\mathrm{int}(S\times[-1,1]))} & \\ \swarrow & & \searrow \\ \dfrac{\pi_1(M)}{\Gamma_{k+1}\pi_1(M)} & \overset{\exists!}{\underset{\simeq}{\dashrightarrow}} & \dfrac{\pi_1(M_S)}{\Gamma_{k+1}\pi_1(M_S)}. \end{array}$$

By composing it with ψ, we obtain a parametrization

$$\psi_S : G \xrightarrow{\simeq} \pi_1(M_S)/\Gamma_{k+1}\pi_1(M_S)$$

of the k-th nilpotent quotient of $\pi_1(M_S)$. In order to compare $\mu_k(M,\psi)$ with $\mu_k(M_S,\psi_S)$, we consider the mapping torus of s

$$\mathrm{t}(s) := (S \times [-1,1]\,/\!\sim) \cup \left(S^1 \times D^2\right).$$

Here, the equivalence relation $\sim$ identifies $s(x)\times 1$ with $x \times (-1)$, and the meridian $1\times\partial D^2$ of the solid torus $S^1\times D^2$ is glued along the circle $*\times[-1,1]\,/\!\sim$ (where $* \in \partial S$) while the longitude $S^1 \times 1$ is glued along $\partial S \times 1$. We note that $\mathrm{t}(s)$ is a closed connected oriented 3-manifold, and that the inclusion $S = S \times 1 \subset \mathrm{t}(s)$ defines an isomorphism

$$\varphi_s : \pi_1(S)/\Gamma_{k+1}\pi_1(S) \xrightarrow{\simeq} \pi_1(\mathrm{t}(s))/\Gamma_{k+1}\pi_1(\mathrm{t}(s)).$$

Besides, the inclusion $S \subset M$ defines a homomorphism

$$i : \pi_1(S)/\Gamma_{k+1}\pi_1(S) \longrightarrow \pi_1(M)/\Gamma_{k+1}\pi_1(M).$$

Then, μ_k varies as follows[(1)] under the surgery $M \rightsquigarrow M_S$:

$$\mu_k(M_S, \psi_S) - \mu_k(M, \psi) = \psi_*^{-1} i_*\big(\mu_k(\mathrm{t}(s), \varphi_s)\big) \in H_3(G; \mathbb{Z}). \tag{0.1}$$

This variation formula suggests the following construction, relative to a compact connected oriented surface Σ with one boundary component. Let $\mathcal{I}(\Sigma)$ be the Torelli group of Σ and let $\pi := \pi_1(\Sigma, *)$ be the fundamental group of Σ, where $* \in \partial\Sigma$. Let

$$\mathcal{I}(\Sigma) = \mathcal{I}(\Sigma)[1] \supset \mathcal{I}(\Sigma)[2] \supset \mathcal{I}(\Sigma)[3] \supset \cdots$$

be the *Johnson filtration* of $\mathcal{I}(\Sigma)$, whose k-th subgroup $\mathcal{I}(\Sigma)[k]$ consists of (the isotopy classes of) the homeomorphisms $s : \Sigma \to \Sigma$ that act trivially at the level of $\pi/\Gamma_{k+1}\pi$. The previous discussion shows that the map

$$M_k : \mathcal{I}(\Sigma)[k] \longrightarrow H_3\left(\pi/\Gamma_{k+1}\pi; \mathbb{Z}\right), \ \ s \longmapsto \mu_k(\mathrm{t}(s), \varphi_s)$$

plays a crucial role in the study of nilpotent homotopy types, and formula (0.1) shows that it is a group homomorphism. The homomorphism M_k has been studied by Heap in [17]. By considering the simplicial model of $\mathrm{K}\left(\pi/\Gamma_{k+1}\pi, 1\right)$, he proves that M_k is equal to Morita's refinement of the k-th Johnson homomorphism, whose definition is purely algebraic and involves the bar complex of a group [32]. Thus, in the sequel, we will refer to M_k as the *k-th Morita homomorphism.*

Since Lie algebra homology is simpler than group homology, one would like to replace the group $\pi/\Gamma_{k+1}\pi$ by its Malcev Lie algebra $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ in the above discussion. Thus, one defines a Lie analogue of the k-th Morita homomorphism

$$m_k : \mathcal{I}(\Sigma)[k] \longrightarrow H_3\big(\mathfrak{m}\left(\pi/\Gamma_{k+1}\pi\right); \mathbb{Q}\big)$$

by imitating Morita's original definition of M_k [32], the bar complex of a group being simply replaced by the Koszul complex of its Malcev Lie algebra. The homomorphism m_k and, in particular its relationship with the theory of finite-type invariants, is the main subject of this paper whose contents we now describe.

First of all, let us recall that the Lie algebra $\mathfrak{m}\left(\pi/\Gamma_{k+1}\pi\right)$ is free nilpotent of class k. More precisely, if we set $H := H_1(\Sigma; \mathbb{Q})$ and if we denote by $\mathfrak{L}(H)$ the free Lie algebra generated by H, then we have a (non-canonical) isomorphism

$$\mathfrak{L}(H)/\Gamma_{k+1}\mathfrak{L}(H) \simeq \mathfrak{m}\left(\pi/\Gamma_{k+1}\pi\right). \tag{0.2}$$

[(1)] This can be proved by a simple homological computation in a singular 3-manifold that contains the three of M, M_S and $\mathrm{t}(s)$. Similar formulas are shown in [11, Theorem 2] and [17, Theorem 5.2] by cobordism arguments.