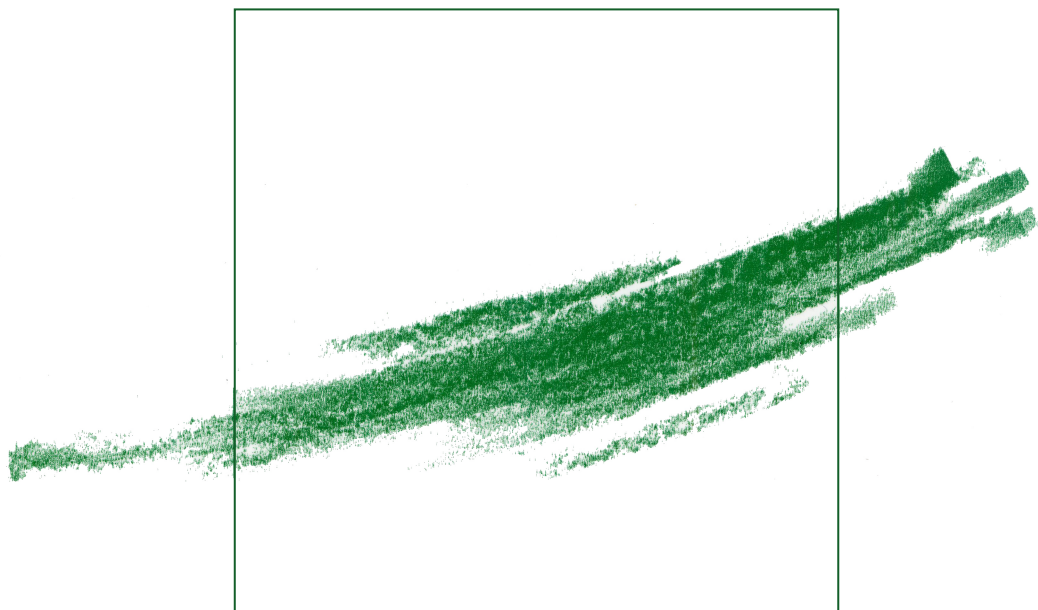


COURS SPÉCIALISÉS
COLLECTION SMF

Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23

**INTEGRAL GEOMETRY FROM BUFFON
TO GEOMETERS OF TODAY**

Rémi Langevin

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CONTENTS

Introduction	7
1. The beginning of integral geometry	9
1.1. Buffon	9
1.2. The birth of the notion of geometric measure	11
2. The Euclidean plane	15
2.1. Geometric measures on sets of lines	15
2.2. The Gauss map	20
3. Two dimensional convex bodies and translations	27
3.1. Envelopes	27
3.2. Support functions and hérissons	29
3.3. Minkowski sum and mixed volumes	34
4. Blaschke's formulas and kinematic formulas	39
4.1. Poincaré's formulas	39
4.2. Blaschke's formulas	41
5. Grassmann manifolds and flag spaces	45
5.1. Grassmann manifolds	45
5.2. Invariant metrics and measures	48
5.3. Flags	49
6. Integral geometry of polyhedral surfaces in \mathbb{R}^3	51
6.1. Theorema egregium	52
6.2. Gauss-Bonnet theorem	53
6.3. Projections on lines	54
6.4. Global integral geometry of polyhedral surfaces	55
7. Surfaces and curves in space	57
7.1. Cauchy-Crofton formulas in \mathbb{R}^3	57
7.2. The Gauss map and the principal curvatures	58
7.3. Total curvature of curves in \mathbb{R}^3	63
8. Integral geometry and topology	67
8.1. Critical points and Gauss curvature, Chern and Lashof's theorem	67

8.2. Total curvature of closed curves and knots	68
8.3. More theorems involving the topology of an immersion or an embedding	69
9. Tight immersions	75
9.1. Definition	75
9.2. Plane curves and planar domains	76
9.3. Surfaces	79
10. 3-dimensional convex bodies and related matters	85
10.1. Support function	85
10.2. Quermassintegrals and Steiner's formula	86
10.3. Orthogonal projections, polar varieties, and length of surfaces of \mathbb{R}^3 ...	90
10.4. Tubes (2)	92
10.5. Localization of the length L_1	97
10.6. Variation of a functional, linear kinematic formulas	100
11. Integral geometry in spheres	107
11.1. The spherical formula of Cauchy and Crofton	107
11.2. Flags	109
12. Integral geometry of foliations	117
12.1. Codimension 1 foliations of a domain of \mathbb{R}^3	117
12.2. Foliations of surfaces of constant curvature	130
12.3. Codimension one foliations of spaces of constant curvature in dimension 3	142
12.4. Tight foliations	143
12.5. Foliations of codimension higher than one	145
12.6. Diverging integrals	146
13. Integral geometry in Lorentz spaces	147
13.1. A Cauchy formula in the Lorentz plane	148
13.2. Other Lorentz manifolds	149
14. The space of spheres	153
14.1. Lorentz-Minkowski space and spheres	153
14.2. Spheres of dimension 0	156
14.3. Points and circles in \mathbb{S}^2	159
14.4. Spheres of dimension two in \mathbb{S}^3	162
15. Integral geometry in hyperbolic spaces	169
15.1. Measure on the set of geodesics of \mathbb{H}^2	169
15.2. Integral geometry of boundaries of compact convex domains and surfaces	172
15.3. Integral geometry in compact hyperbolic surfaces	175
15.4. Integral geometry with horocycles	176
16. The spherical analogue of tightness	179

16.1. The circle two-piece property	179
16.2. The spherical two-piece property	181
17. Conformal integral geometry	187
17.1. Some conformal invariants of a knot or a link	187
17.2. Spheres and two-component links	191
17.3. A conformal (Fary-Fenchel-Milnor)-like theorem	199
17.4. Intersection of surfaces of the sphere \mathbb{S}^3 with spheres	206
18. Conformal integral geometry of foliations	209
18.1. A local conformal invariant of 3-dimensional foliation of \mathbb{R}^3	209
18.2. Local and bilocal invariants of plane foliations	212
19. Complex integral geometry	219
19.1. Set of lines	220
19.2. Cauchy-Crofton formula in \mathbb{C}^2	221
19.3. Local geometry of complex curves in \mathbb{C}^2	222
19.4. Critical points of projections on complex lines and the complex Gauss map	224
20. Appendix	235
20.1. Archimedes computation of an area	235
20.2. Measure associated with a Riemannian metric	237
20.3. Curvature associated to a Riemannian metric on a surface	237
20.4. The Brouwer degree of a map and index of an isolated singularity	238
20.5. Critical points of a function	239
20.6. Generic properties	240
20.7. Coarea formula	240
20.8. Foliations	242
20.9. Fiber bundles	245
20.10. Hopf fibration	246
20.11. Linear pencils	247
20.12. Models of the hyperbolic plane	250
20.13. Dupin cyclides	252
20.14. Complements and conjectures in conformal geometry	254
20.15. Topology	259
20.16. The Poincaré-Hopf theorem; an extension for non-orientable foliations of the disc	263
Index of notations	265
Index of names	267
Index	269
Bibliography	273

