Revue d'histoire des mathématiques,
2 (1996), p. 67–93.

FROM ATTRACTION THEORY TO EXISTENCE PROOFS: THE EVOLUTION OF POTENTIAL-THEORETIC METHODS IN THE STUDY OF BOUNDARY-VALUE PROBLEMS, 1860–1890

Thomas ARCHIBALD (*)

ABSTRACT. — This paper examines developments in the study of boundary-value problems between about 1860 and 1890, in the context of the general evolution of this theory from the physical models in which the subject has its roots to a free-standing part of pure mathematics. The physically-motivated work of Carl Neumann and his method of the arithmetic mean appear as an initial phase in this development, one which employs physical models as an integral part of its reasoning and which concentrates on geometrical hypotheses concerning the regions under study. The alternating method of Hermann Amandus Schwarz, roughly contemporary to that of Neumann, exhibits more strongly the analytic influence of Weierstrass. Both methods form the essential background to Émile Picard's method of successive approximations, developed by him following a reading of both men's work. Picard's work, analytically rigorous and remote from physical argument, marks both a transition of the subject matter from applied to pure mathematics, and the full comprehension and mastery of Weierstrassian methods in the French context.

RÉSUMÉ. — DE LA THÉORIE DE L'ATTRACTION AUX THÉORÈMES D'EXISTENCE : L'ÉVOLUTION DES MÉTHODES DE LA THÉORIE DU POTENTIEL DANS L'ÉTUDE DES PROBLÈMES AUX LIMITES, 1860–1890. Cet article analyse les contributions à l'étude des problèmes aux limites, au cours des années 1860–1890, dans le contexte de l'évolution générale de la théorie qui, partant des modèles physiques où la question trouve ses racines, se constitue en domaine autonome relevant des mathématiques pures. Les travaux de Carl Neumann inspirés par la physique et sa méthode de la moyenne apparaissent comme la phase initiale de cette évolution, celle qui emploie des modèles physiques comme partie intégrante des raisonnements et qui se centre sur les hypothèses géométriques relatives aux régions considérées. Le procédé alterné dû à Hermann Amandus Schwarz, méthode à peu près contemporaine, porte nettement la marque de l'analyse weierstrassienne. Ces deux méthodes constituent pour l'essentiel le fonds où s'inscrira la méthode des approximations successives d'Émile Picard, que celui-ci a développée à la suite de la lecture des travaux des deux auteurs précédents. Les recherches de Picard, analytiquement rigoureuses et éloignées des argumentations

(*) Texte reçu le 4 septembre 1995, révisé le 8 février 1996.
Thomas ARCHIBALD, Department of Mathematics and Statistics, Acadia University, Wolfville, Nova Scotia, BOP 1X0 Canada. E.-mail: tom.archibald@acadiau.ca.

physiques, marquent à la fois le passage du domaine des mathématiques appliquées à celui des mathématiques pures et l'avènement de la pleine compréhension et maîtrise des méthodes de Weierstrass en France.

1. INTRODUCTION

On July 6, 1937, Émile Picard was awarded the *Prix Mittag-Leffler* at the Institut de France. The prize was awarded by the Institut Mittag-Leffler for "*les découvertes qui constituent une source nouvelle et importante de progrès futurs pour les Sciences mathématiques*", and consisted of a gold medal with the portrait of the winner, a diploma, and a personalized set of *Acta mathematica*. At the ceremony, Picard recounted the fame of Karl Weierstrass and his Swedish disciple Gösta Mittag-Leffler in the Paris of the mid-1880s:

"*Il arriva même que dans une de ces cérémonies, dites les Ombres, où les Polytechniciens font d'innocentes plaisanteries sur leurs professeurs, on annonça la découverte d'un nouveau verset de la* Genèse*, où il était écrit*: *'Dieu créa Weierstrass, puis, ne trouvant pas bon que Weierstrass fût seul, il créa Mittag-Leffler'*" [Picard 1938, pp. xxiii–xxiv].

The joke shows a widespread appreciation of the importance of Weierstrassian analysis in the French mathematics of the time, particularly those aspects of it most closely associated with the name of Mittag-Leffler: the theory of functions of a complex variable and its applications to other areas of analysis. Of course, France could lay claim to much of this theory thanks to the foundational work of Cauchy. Its later elaborations in Germany, due to Riemann and Weierstrass among others, had become known to the French mathematical community largely through the intermediary of Hermite, who lectured on these matters to Picard among others. In the next generation, Picard himself was instrumental in introducing these German techniques to French mathematicians and students, and so was an important figure in the development of an international style of mathematics from a congeries of distinct national schools. The requirements of Weierstrassian rigour, particularly in analysis, were instrumental in this transition. Originally conceived as a language of justification, Weierstrass's analysis soon revealed itself to be a powerful tool for discovery as well; and this feature in part accounts for its success among his students and adherents, as well as for its spread to mathematical communities outside Germany. In that context, it was natural that existence and uniqueness theory for partial differential equations should assume a front-line position.

Until the mid-nineteenth century, partial differential equations were not studied in a unified fashion, and there were few general results which could be considered to unify the theory. For the most part, individual equations were studied in the context where they arose; in the case of boundary-value problems, this meant that the Laplace-Poisson equation was studied in connection with the theory of gravitation, electrostatics, or steady-state heat conduction, while the wave equation arose in acoustics and optics, etc. The question of existence theorems for boundary-value problems was raised by the well-known critique by Weierstrass of Riemann's justification of the Dirichlet principle, which the latter had employed to show the existence of a solution to the Dirichlet problem for plane regions, given appropriate boundary conditions. The efforts to rehabilitate Riemann's proof were many. The first to succeed, beginning around 1870, were those of Carl Neumann — known as the method of the arithmetic mean, which established the existence of solutions for the Dirichlet problem by a method of approximate solutions; and those of Hermann Amandus Schwarz.

Both Neumann's work and that of Schwarz were seen by most readers as part of a specialty, called potential theory, which concerned itself not only with the theory of the Laplace-Poisson equation and associated boundary-value problems, but also with the associated special functions (spherical harmonics, etc.) and with applications especially in gravitation (attractions of ellipsoids, figures of planets) and electromagnetic theory (equilibrium electrostatic densities given an external force, forces given densities, etc.). However, the work of Neumann and Schwarz was generalized, in the hands of Emile Picard, to become the method of successive approximations, which Picard showed could be applied to a wide variety of boundary-value problems for second-order equations. At around the same time, Picard's Paris colleague Henri Poincaré began to systematically investigate the analogies between the various partial differential equations, mostly of second order, which are associated with physical problems.

These simultaneous efforts may be seen as part of the establishment of the subject of partial differential equations as a recognized research specialty, independent of its applications. At the same time, to an increasing degree, mathematical physics and pure mathematics were in the process of disciplinary separation. Hence fewer mathematicians undertook both kinds of research, and an increased specialization of institutions (such as journals, university departments and institutes) also occurred. This in turn led to a lessened emphasis on direct physical applications in potential theory, and to the subsuming of the latter into partial differential equations as a research specialty. We may see that this in a way completes the divorce of potential theory from physics, though of course certain problems were still of interest to physicists. These would however then be seen as applications of the theory, rather than as instances of it, and tended to be undertaken by different individuals from the pure mathematical problems.

It is the purpose of this paper to examine aspects of this transition. In particular, we shall concentrate on the background to the development of the method of successive approximations by Picard. As Lützen has discussed in detail, the method was used as early as 1830 by Liouville [Lützen 1990, pp. 447–448], though more as a solution method than as an existence proof, which is how Picard employs it. The pivotal position of the so-called Dirichlet problem in these developments makes it convenient to begin with a discussion of research related to this question.

2. CARL NEUMANN AND THE DIRICHLET PROBLEM

The Dirichlet problem is the following: given the values of a function on the boundary of a region in space or in the plane, find a function which is harmonic on the region and which takes on those boundary values. It is closely associated with the conformal mapping question; for if we can solve the problem for a particular region (*e.g.* a circular disc) we can extend the solution to other regions through composition with a harmonic function which provides a conformal representation of the region onto the disc. This idea was first worked out by Bernhard Riemann in his 1851 dissertation. There were well-known difficulties with Riemann's approach, however; his existence proof depended on the "Dirichlet principle", about which much has been written. In particular, Weierstrassian critiques called into question the validity of the Riemann mapping theorem, one of the cornerstones of Riemann's function theory.

These critiques were addressed, from rather different standpoints, by Carl Neumann (1832–1925) and Hermann Amandus Schwarz (1843–1921), beginning in the 1860s and culminating in successful results about 1870. These works are indicative of the transitional state of affairs with regard to partial differential equations in Germany at the time, and were of particular importance to Picard.

Carl Neumann's earliest work on potential theory had revolved around the Dirichlet problem; his other interests in the period show that he was influenced by Riemann in this regard. In 1861, while at Halle, Neumann produced the first of his many papers on this question, The paper, "Ueber die Integration der partiellen Differentialgleichung $\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0$", treated the Dirichlet problem in the plane. It contains two principal results, both solving the problem explicitly for a limited class of regions. For Neumann, the work remained close to his physical investigations; he began by pointing out the analogy with the three-dimensional problem of steady state temperature distribution. The problem, specifically, is to find a function $F(x, y)$ which satisfies the Laplace equation inside a connected region R in the plane bounded by a curve of arbitrary form such that F and its first derivatives remain finite, single-valued, and continuous inside R, possessing given values on the boundary of R. In the three-dimensional case, Neumann points out, the use of the theory developed by Green and Gauss of the potential corresponding to the Newtonian attraction law is of great assistance with the problem, and further:

"*Likewise it is useful here in considering our planar problem to assume as an auxiliary a hypothetical matter or fluid which is distributed arbitrarily in the* plane*, for which the potential of two particles on one another is equal to the product of their masses multiplied by the logarithm of the distance between them.*"[1]

That one requires auxiliary fluids rather than auxiliary functions seems

[1] "*Ebenso ist es hier bei Behandlung unseres Problems der Ebene zweckmässig, eine fingirte Materie oder ein fingirtes Fluidum zu Hülfe zu nehmen, welches auf beliebige Weise in der* Ebene *vertheilt wird, und für welches das Potential zweier Theilchen aufeinander gleich ist dem Product ihrer Massen multiplicirt mit dem* Logarithmus ihrer Entfernung*"* [Neumann 1861, p. 336].