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IND-SHEAVES

Masaki Kashiwara, Pierre Schapira

Abstract. — Sheaf theory is not well suited to the study of various objects in Analysis which are not defined by local properties. The aim of this paper is to show that it is possible to overcome this difficulty by enlarging the category of sheaves to that of ind-sheaves, and by extending to ind-sheaves the machinery of sheaves.

Let X be a locally compact topological space and let k be a commutative ring. We define the category $\mathrm{I}(k_X)$ of ind-sheaves of k-modules on X as the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves of k-modules on X with compact support, and we construct "Grothendieck's six operations" in the derived categories of ind-sheaves, as well as new functors which naturally arise.

A method for constructing ind-sheaves is the use of Grothendieck topologies associated with families $\mathcal{T}$ of open subsets satisfying suitable properties. Sheaves on the site $X_{\mathcal{T}}$ naturally define ind-sheaves.

When X is a real analytic manifold, we consider the subanalytic site X_{sa} associated with the family of open subanalytic subsets, and construct various ind-sheaves by this way. We obtain in particular the ind-sheaf $\mathcal{C}^{\infty,t}_X$ of tempered C^∞-functions, the ind-sheaf $\mathcal{C}^{\infty,w}_X$ of Whitney C^∞-functions and the ind-sheaf $\mathcal{D}b^t_X$ of tempered distributions. On a complex manifold X, we concentrate on the study of the ind-sheaf $\mathcal{O}^t_X$ of "tempered holomorphic functions" and prove an adjunction formula for integral transforms in this framework.

***Résumé* (Ind-faisceaux). —** La théorie des faisceaux n'est pas bien adaptée à l'étude de divers objets de l'Analyse qui ne sont pas définis par des propriétés locales. Le but de cet article est de montrer que l'on peut surmonter cette difficulté en élargissant la catégorie des faisceaux à celle des ind-faisceaux, et étendre à ceux-ci le formalisme des faisceaux.

Soit X un espace localement compact et soit k un anneau commutatif. Nous définissons la catégorie $\mathrm{I}(k_X)$ des ind-faisceaux de k-modules sur X comme la catégorie des ind-objets de la catégorie $\mathrm{Mod}^c(k_X)$ des faisceaux de k-modules sur X à support compact, et nous construisons les « six opérations de Grothendieck » dans la catégorie dérivée des ind-faisceaux, ainsi que de nouveaux foncteurs qui apparaissent naturellement.

Une méthode pour construire des ind-faisceaux est l'utilisation de topologies de Grothendieck associées à des familles $\mathcal{T}$ d'ouverts de X satisfaisant certaines propriétés. Les faisceaux sur le site $X_{\mathcal{T}}$ définissent alors naturellement des ind-faisceaux.

Quand X est une variété analytique, nous considérons le site sous-analytique X_{sa} associé à la famille des ouverts sous-analytiques et nous construisons ainsi divers ind-faisceaux. Nous obtenons en particulier le ind-faisceau $\mathcal{C}_X^{\infty,t}$ des fonctions C^∞ tempérées, le ind-faisceau $\mathcal{C}_X^{\infty,w}$ des fonctions C^∞ de type Whitney, et le ind-faisceau $\mathcal{D}b_X^t$ des distributions tempérées.

Sur une variété complexe X, nous concentrons notre étude sur le ind-faisceau $\mathcal{O}_X^t$ des « fonctions holomorphes tempérées » et prouvons une formule d'adjonction dans ce cadre.

CONTENTS

INTRODUCTION

Sheaf theory is not well suited to the study of various objects in Analysis which are not defined by local properties, such as for example holomorphic functions with tempered growth. The aim of this paper is to show that it is possible to overcome this difficulty by enlarging the category of sheaves to that of ind-sheaves, and by extending to ind-sheaves the machinery of sheaves.

Recall that if $\mathcal{C}$ is an abelian category, the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of $\mathcal{C}$ has many remarkable properties: it contains $\mathcal{C}$ and admits small inductive limits, it is abelian and the natural functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is exact and fully faithful. Moreover $\mathrm{Ind}(\mathcal{C})$ is, in a certain sense, "dual" to $\mathcal{C}$.

For a locally compact topological space X and a commutative ring k, we introduce the category $\mathrm{I}(k_X)$ of ind-sheaves of k-modules on X as the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves of k-modules with compact support in X. This construction has some analogy with that of distributions: the space of distributions is bigger than that of functions, and is dual to the space of functions with compact support. This last condition implies the local nature of distributions, and similarly, we prove that ind-sheaves form a stack (a "sheaf of categories").

We construct "Grothendieck's six operations" in the derived categories of ind-sheaves, as well as new functors which naturally arise.

There is a method for constructing ind-sheaves using Grothendieck topologies. We consider on X a family $\mathcal{T}$ of open subsets satisfying suitable properties and associate to it a site. In particular, when X is a real analytic manifold and $\mathcal{T}$ is the family of subanalytic open subsets, we obtain the "subanalytic site X_{sa}". We prove that the category of ind-objects of $\mathcal{T}$-coherent sheaves is equivalent to the category of sheaves on the site $X_{\mathcal{T}}$. Therefore, such sheaves naturally define ind-sheaves.

As already mentioned, ind-sheaves allow us to treat functions with growth conditions in the formalism of sheaves. On a complex manifold X, we can define the ind-sheaf of "tempered holomorphic functions" $\mathcal{O}^t_X$, or the ind-sheaf of "Whitney holomorphic functions" $\mathcal{O}_X^{\mathrm{w}}$, and obtain for example the sheaves of distributions or of C^∞-functions using Sato's construction of hyperfunctions, simply replacing $\mathcal{O}_X$ with $\mathcal{O}_X^t$ or $\mathcal{O}_X^{\mathrm{w}}$. We also prove an adjunction formula for integral transforms in this framework.

The contents of these Notes is as follows.

Chapters I and II are a short review, without proofs, of the theory of ind-objects with some applications to derived categories, and the theory of sheaves on Grothendieck topologies. Of course all these theories (invented by Grothendieck) are now classical. However, we shall also recall some technical statements extracted from **[13]** which are new.

Chapter III is devoted to stacks on a locally compact space X. We introduce the notion of a proper stack, show that this notion is stronger than the usual one of a stack, although its axioms are quite easy to check, and prove that the indization of a proper stack is a proper stack. There are new functors: ι from a stack to the associated ind-stack, and its left inverse α. Under reasonable conditions which will be satisfied by sheaves, α also admits a left adjoint β.

Ind-sheaves are introduced in Chapter IV, in which we first construct the internal operations: tensor product denoted by $\otimes$, and internal hom denoted by $\mathcal{I}hom$. We then construct the external operations: inverse image f^{-1}, direct image f_* and proper direct image $f_{!!}$. Finally, we study the various relations among all these functors. Note that the proper direct image of a sheaf is not the same in general whether we calculate it in sheaf theory or ind-sheaf theory.

In Chapter V, we derive all the functors we have constructed, and give relations among the derived functors. Moreover, as in the classical case, the functor $Rf_{!!}$ admits a right adjoint $f^!$, and we study its main properties. One of the difficulties of this study is that the category of ind-sheaves does not have enough injective objects. In this chapter, we also introduce the notions of ind-sheaves of rings and modules. This will be necessary for applications. For example, the ind-sheaf $\mathcal{O}_X^t$ of "tempered holomorphic functions" cannot be defined in the derived category of ind-sheaves of $\mathcal{D}_X$-modules, and one has to replace $\mathcal{D}_X$ with the ind-sheaf of rings $\beta_X(\mathcal{D}_X)$. As we shall see, this does not cause much trouble.

Chapter VI is devoted to the construction of ind-sheaves using Grothendieck topologies. We consider a family $\mathcal{T}$ of open subsets of X satisfying suitable properties, and its subfamily $\mathcal{T}_c$ of relatively compact open sets. We define the category $\mathrm{Coh}(\mathcal{T}_c)$ as the full subcategory of $\mathrm{Mod}(k_X)$ consisting of cokernels of morphisms $F \to G$ with F and G finite sums of sheaves of the type k_{XU} with $U \in \mathcal{T}_c$ and prove that this category is abelian. Then we define the site $X_\mathcal{T}$ whose family of objects is $\mathcal{T}$, a covering of $U \in \mathcal{T}$ being a *locally finite covering in* X. We study the category $\mathrm{Mod}(k_\mathcal{T})$ of sheaves on this site and prove that it is equivalent to the category of ind-objects of $\mathrm{Coh}(\mathcal{T}_c)$. Hence, there is a natural fully faithful exact functor from $\mathrm{Mod}(k_\mathcal{T})$ to

$\mathrm{I}(k_X)$, and this is a useful tool for constructing ind-sheaves. The category $\mathrm{I}(k_X)$ of ind-sheaves on X is much bigger than the category $\mathrm{Mod}(k_{\mathcal{T}})$. The first one does not have enough injectives, which make the theory rather difficult, unlike the second one. On the other hand the natural functor from $\mathrm{Mod}(k_X)$ to $\mathrm{I}(k_X)$ is exact, which fails when we replace $\mathrm{I}(k_X)$ with $\mathrm{Mod}(k_{\mathcal{T}})$.

We apply these results in Chapter VII and obtain the "subanalytic site" X_{sa} on a real analytic manifold X by taking the family of open subanalytic subsets as $\mathcal{T}$. We construct various ind-sheaves on this site, and when X is a complex manifold this allows us to define in particular the ind-sheaves $\mathcal{O}_X^t$ and $\mathcal{O}_X^{\mathrm{w}}$ of "tempered holomorphic functions" and "Whitney holomorphic functions", respectively. We prove formulas for direct images, inverse images and composition with a regular holonomic kernel for the ind-sheaf $\mathcal{O}_X^t$ (in the framework of $\mathcal{D}$-modules), from which we deduce a general adjunction formula for integral transforms.