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# POLARIZABLE TWISTOR $\mathscr{D}$-MODULES

**Claude Sabbah**

***Abstract.* —** We prove a Decomposition Theorem for the direct image of an irreducible local system on a smooth complex projective variety under a morphism with values in another smooth complex projective variety. For this purpose, we construct a category of polarized twistor $\mathscr{D}$-modules and show a Decomposition Theorem in this category.

***Résumé* ($\mathscr{D}$-modules avec structure de twisteur polarisable). —** Nous montrons un théorème de décomposition pour l'image directe d'un système local irréductible sur une variété projective complexe lisse par un morphisme à valeurs dans une autre variété projective complexe lisse. À cet effet, nous construisons une catégorie de $\mathscr{D}$-modules avec structure de twisteur polarisée et nous montrons un théorème de décomposition dans cette catégorie.

# CONTENTS

# INTRODUCTION

Let $X$ be a smooth complex projective manifold and let $\mathscr{F}$ be a locally constant sheaf of $\mathbb{C}$-vector spaces of finite dimension on $X$. We assume that $\mathscr{F}$ is *semisimple*, *i.e.*, a direct sum of irreducible locally constant sheaves on $X$. Then it is known that, given any ample line bundle on $X$, the corresponding Hard Lefschetz Theorem holds for the cohomology of $X$ with values in $\mathscr{F}$: if $\mathscr{F}$ is constant, this follows from Hodge theory; for general semisimple local systems, this was proved by C. Simpson [**63**] using the existence of a harmonic metric [**13**]. The existence of such a metric also allows him to show easily that the restriction of $\mathscr{F}$ to any smooth subvariety of $X$ remains semisimple.

In this article, we extend to such semisimple local systems other properties known to be true for the constant sheaf, properties usually deduced from Hodge theory. These properties will concern the behaviour with respect to morphisms. They were first proved for the constant sheaf (*cf.* [**15, 17, 60, 66, 3, 28**]) and then, more generally, for local systems underlying a polarizable Hodge Module, as a consequence of the work of M. Saito [**56**].

Given a local system $\mathscr{F}$ of finite dimensional $\mathbb{C}$-vector spaces on a complex manifold $X$, it will be convenient to denote by ${}^p\mathscr{F}$ the associated perverse complex $\mathscr{F}[\dim X]$, *i.e.*, the complex having $\mathscr{F}$ as its only nonzero term, this term being in degree $-\dim X$.

The proof of the following results will be given in § 6.1.

***Main Theorem 1* (Decomposition Theorem).** *Let $X$ be a smooth complex projective variety and let $\mathscr{F}$ be a semisimple local system of finite dimensional $\mathbb{C}$-vector spaces on $X$. Let $U$ be an open set of $X$ and let $f : U \to Y$ be a proper holomorphic mapping in a complex manifold $Y$. Fix an ample line bundle on $X$. Then*

(1) *the relative Hard Lefschetz Theorem holds for the perverse cohomology sheaves ${}^p\mathscr{H}^i(\boldsymbol{R}f_*{}^p\mathscr{F}_{|U})$ of the direct image;*

(2) *the direct image complex $\boldsymbol{R}f_*{}^p\mathscr{F}_{|U}$ decomposes (maybe non canonically) as the direct sum of its perverse cohomology sheaves:*

$$\boldsymbol{R}f_*{}^p\mathscr{F}_{|U} \simeq \bigoplus_i {}^p\mathscr{H}^i(\boldsymbol{R}f_*{}^p\mathscr{F}_{|U})[-i];$$

(3) *each perverse cohomology sheaf* ${}^p\mathscr{H}^i(\boldsymbol{R}f_*{}^p\mathscr{F}_{|U})$ *decomposes as the direct sum of intersection complexes supported on closed irreducible analytic subsets* $Z$ *of* $Y$, *i.e., of the form* $\mathrm{IC}^\bullet({}^p\mathscr{L})$, *where* $\mathscr{L}$ *is a local system on a smooth open dense set* $Z \smallsetminus Z'$, *with* $Z'$ *closed analytic in* $Z$;

(4) *if moreover* $U = X$ *and* $Y$ *is projective, then each perverse cohomology sheaf* ${}^p\mathscr{H}^i(\boldsymbol{R}f_*{}^p\mathscr{F})$ *is* semisimple, *i.e., the local systems* $\mathscr{L}$ *are semisimple.*

***Main Theorem 2* (Vanishing cycles).** — *Let* $X$ *be a smooth complex projective variety and let* $\mathscr{F}$ *be a semisimple local system on* $X$. *Let* $U$ *be an open set of* $X$ *and let* $f : U \to \mathbb{C}$ *be a holomorphic function on* $U$ *which is proper. Then, for any* $\ell \in \mathbb{Z}$, *the perverse complexes* $\mathrm{gr}_\ell^{\mathrm{M}}{}^p\psi_f{}^p\mathscr{F}$ *and* $\mathrm{gr}_\ell^{\mathrm{M}}{}^p\phi_f{}^p\mathscr{F}$, *obtained by grading with respect to the monodromy filtration the perverse complexes of nearby or vanishing cycles, are semisimple perverse sheaves on* $f^{-1}(0)$.

***Remarks***

(1) We note that (1) $\Rightarrow$ (2) in Main Theorem 1 follows from an argument of Deligne [**15**].

(2) The nearby and vanishing cycles functors $\psi_f$ and $\phi_f$ defined by Deligne [**19**] are shifted by $-1$, so that they send perverse sheaves to perverse sheaves. They are denoted by ${}^p\psi_f$ and ${}^p\phi_f$, following M. Saito [**56**].

(3) It is known that the Main Theorem 1 implies the *local invariant cycle theorem* for the cohomology with coefficients in $\mathscr{F}$ (*cf.* [**3**, Cor. 6.2.8 and 6.2.9], see also [**57**, Cor. 3.6 and 3.7]). If for instance $Y = \mathbb{C}$ then, for any $k \geqslant 0$ and for $t \neq 0$ small enough, there is an exact sequence

$$H^k(f^{-1}(0), \mathscr{F}) \longrightarrow H^k(f^{-1}(t), \mathscr{F}) \xrightarrow{T - \mathrm{Id}} H^k(f^{-1}(t), \mathscr{F}),$$

where $T$ denotes the monodromy. It also implies the exactness of the *Clemens-Schmid sequence*.

(4) Owing to the fact that, if $\mathscr{F}_\mathbb{Q}$ is a perverse complex of $\mathbb{Q}$-vector spaces on a complex analytic manifold, then $\mathscr{F}_\mathbb{Q}$ is semisimple if and only if $\mathscr{F}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Q} \mathscr{F}_\mathbb{Q}$ is so, the previous results apply as well to $\mathbb{Q}$-local systems, giving semisimple $\mathbb{Q}$-perverse complexes as a result.

(5) It would be possible to define a category of perverse complexes "of smooth origin", obtained after iterating various operations starting from a semisimple local system on a smooth complex projective variety, *e.g.*, taking perverse cohomology of a projective direct image, taking monodromy-graded nearby or vanishing cycles relative to a projective holomorphic function, taking sub-quotients of such objects. The perverse complexes in this category are semisimple.

(6) A conjecture of M. Kashiwara [**34**] —which was the main motivation for this work— asserts in particular that these results should hold when $\mathscr{F}$ is any semisimple *perverse sheaf* (with coefficients in $\mathbb{C}$) on $X$. In the complex situation that we consider, they are proved when $\mathscr{F}$ underlies a *polarizable Hodge Module*, *i.e.*, if on a smooth

dense open set of its support, the perverse sheaf $\mathscr{F}$ is (up to a shift) a local system defined over $\mathbb{Q}$ or $\mathbb{R}$ underlying a variation of polarized Hodge structures defined over $\mathbb{Q}$ or $\mathbb{R}$: this is a consequence of the work of M. Saito [**56, 58**] and [**12, 35**], and of the known fact (see [**21**]) that, on a smooth Zariski open set of a projective variety, the local system underlying a variation of complex Hodge structures is semisimple.

Let us indicate that the conjecture of Kashiwara is even more general, as it asserts that analogues of such results should be true for semisimple holonomic $\mathscr{D}$-modules on smooth complex projective varieties. However, we will not seriously consider non regular $\mathscr{D}$-modules in this article.

(7) First were proved the arithmetic analogues of these theorems, *i.e.*, for "pure sheaves" instead of semisimple sheaves (*cf.* [**3**]) and they were used to give the first proof of the Decomposition Theorem for the constant sheaf in the complex case. An arithmetic approach to the conjecture of Kashiwara (at least for $\mathbb{C}$-perverse sheaves) has recently been proposed by V. Drinfeld [**24**].

(8) It should be emphasized that we work with *global* properties on a projective variety, namely, semisimplicity. Nevertheless, the main idea in the proof is to show that these global properties can be expressed by *local* ones, *i.e.*, by showing that each irreducible local system on $X$ underlies a variation of some structure, analogous to a polarized Hodge structure, called a *polarized twistor structure.* Extending this to irreducible perverse sheaves is the contents of Conjecture 4.2.13.

(9) It will be more convenient to work with the category of regular holonomic $\mathscr{D}_X$-modules instead of that of $\mathbb{C}$-perverse sheaves on $X$. It is known that both categories are equivalent *via* the de Rham functor, and that this equivalence is compatible with the corresponding direct image functors or with the nearby and vanishing cycles functors. We will freely use this compatibility.

Let us now give some explanation on the main steps of the proof. We will use three sources of ideas:

(1) the theory of *twistor structures* developed by C. Simpson (after ideas of P. Deligne),

(2) the techniques developed by M. Saito in the theory of polarizable Hodge Modules,

(3) the use of distributions and Mellin transform, as inspired by the work of M. Kashiwara and D. Barlet.

One of the main objectives, when trying to prove a decomposition theorem, is to develop a notion of *weight* satisfying good properties with respect to standard functors. In other words, the category of semisimple local systems (or, better, semisimple perverse sheaves) should satisfy the properties that one expects for pure sheaves. If the Hodge structure contains in its very definition such a notion, it is not clear *a priori* how to associate a weight to an irreducible perverse sheaf: one could give it weight 0, but one should then explain why ${}^p\mathscr{H}^i(\boldsymbol{R}f_*\mathscr{F})$ has weight $i$ for instance. On the other