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INTERSECTION RINGS OF SPACES OF TRIANGLES

Alberto COLLINO and William FULTON

In 1880 Schubert [12] described a space which compactifies the set of (ordered) plane triangles, and described its intersection ring – giving a basis for the cycles in each dimension, and giving algorithms for computing products. In 1954 Semple [13] gave a modern construction of this space, which we denote X, as an algebraic submanifold of a product of projective and Grassmann manifolds. Tyrrell [15] verified Schubert's prescription of the cycles and their relations in codimension one, and calculated a few other intersection products. The aim of this note is to complete this analysis. We give a formula for the Chow ring (or cohomology ring) of this space : it is generated by seven classes in codimension one, with an ideal of relations generated by twelve classes. In particular we verify that Schubert's basis is correct in all dimensions, and the intersections are as he specified. It is interesting, however, that one of the defining relations for the intersection ring is independent of those given by Schubert before he lists the basis.

The proof is remarkably easy. Since the torus of diagonal matrices in $SL(3)$ acts on X with finitely many (72) fixed points, it follows from the work of Bialynicki–Birula [1], [2] that the total Chow group $A^{\cdot}(X)$ of X is free on 72 generators. We define, purely algebraically, a graded ring $A^{\cdot}$ with seven generators and certain relations, and verify that $A^{\cdot}$ has 72 generators – the same basis as given by Schubert. It is easy to verify that there is a homomorphism from the ring $A^{\cdot}$ to the Chow ring $A^{\cdot}(X)$. Since the generator of A^6 maps to the generator of $A^6(X)$, Poincaré duality implies that this homomorphism is an isomorphism.

Because the algorithms for writing any classes in terms of the basic classes are given explicitly, it becomes a simple algebraic exercise to compute any intersection products, and in particular any enumerative formula, involving the basic 72 generators.

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Although modern machinery has often been used to give rigorous proofs of classical formulas in enumerative geometry, this appears to us to be one of the rare instances where a modern framework actually simplifies the classical calculations. Only part of the first few pages of Schubert's calculations appear in this approach. Perhaps the most obscure part of Schubert's paper (pp. 167–181), which may be regarded as a calculation of the Kunneth components of the class of the diagonal on $X \times X$, can be dispensed with, since this is equivalent to knowing the intersection products of all pairs of generators in complementary dimensions.

In this paper we also compute the Chow ring of the space of triangles in a projective bundle over a given variety. This includes the space of triangles in $\mathbb{P}^n$; for $n=3$ a few equations were included at the end of Schubert's paper [Sch]. As he implies, there are few new ideas needed for this generalization ; the present framework makes it quite automatic.

Another approach to the computation of intersections on the space X of plane triangles has been developed by Roberts and Speiser [9], [10]. They show how X can be constructed by starting with $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and forming two blowups, followed by one blowdown. This allows one to work out, although with some difficulty, any intersection products one may wish. That approach requires delving considerably deeper into the geometry of the space X, which is of independent interest. Our approach, on the other hand, gives the whole intersection theory on X all at once, with minimal knowledge needed about its geometry, and no need to verify intersection multiplicities of any but the simplest intersection products.

We were led to this idea by reading the preprint of Ellingsrud and Strømme [5], who used the Bialynicki–Birula theorem to compute the Chow groups of the Hilbert schemes of points in the plane. The simple observation of the present note is that the same theorem will yield the Chow ring of a variety, provided one can guess (say with the help of Schubert !) what the ring should be, and one can produce a suitable homomorphism from this abstract ring to the actual ring.

Le Barz [8] has used Hilbert scheme methods to construct a space of triangles in any non–singular variety. We comment on this in §5.

Schubert gives many applications, of which we discuss only one : to calculate the number of triangles which are simultaneously inscribed in a given plane curve C, and circumscribed about a given plane curve D, assuming C and D are suitably general. Here Schubert makes an error and gives an incorrect formula. This is remarkable not only because of the rarity of any errors in Schubert's formulas, but also because the correct formula had been given a decade earlier by

Caylay [3] ! Schubert's error was not in his discussion of the intersection theory of the space of triangles. Rather, he ignored the fact that the dual of a smooth curve of degree greater than two has singularities. When this is taken into account, the correct formula comes out.

The first section discusses the space X of complete triangles, reviewing that part of the work of Schubert and Semple that we need. The second section is pure algebra, describing the ring $A^{\cdot}$ and giving algorithms for writing any element of $A^{\cdot}$ as a linear combination of 72 basic classes. The proof that $A^{\cdot}$ is the intersection ring of X is given in §3, and the application to inscribed and circumscribed triangles in §4. The extension to higher dimensions, with a few complementary remarks occupies §5. Appendix A contains some algebraic manipulations needed for §2 (and for [12], but Schubert assumed the reader could supply them). Appendix B contains the tables of intersection products of classes of complementary dimensions. In Appendix C we prove a simple "Leray Hirsh" theorem for Chow groups of fibre bundles whose fibre is a variety such as the variety of plane triangles, or any smooth projective variety with $\mathbb{C}^*$ action with finitely many fixed points.

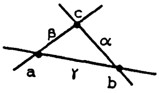
We thank Joe Harris for useful advice about the influence of plane curve singularities on enumerative formulas, and Steven Kleiman for pointing us to Cayley's paper.

Section 1. The compactified space of triangles.

We follow Schubert's notation for ordered triangles in the plane. We sketch a typical member of each type, according to dimension of the loci of such triangles.

A general triangle has vertices a, b, c, with the opposite sides being lines α, β, γ:

Dimension 6



Five–dimensional families :

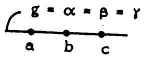
ϵ : the three lines coincide in one line denoted g, on which there are three vertices a, b, c.

τ : dually, the three vertices coincide in a point s, through which pass three lines α, β, γ.

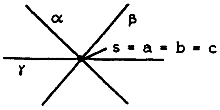
θ_a : the two lines β and γ coincide in a line g, the two points b and c coincide in a point s on g; a is another point on g, while α is another line through s.

θ_b and θ_c are defined similarly, by permuting the vertices and edges.

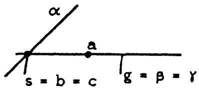
Dimension 5



Type ϵ



Type τ



Type θ_a