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QUASI-ABELIAN CATEGORIES AND SHEAVES

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***Abstract.* —** This memoir is divided in three parts. In the first one, we introduce the notion of quasi-abelian category and link the homological algebra of these categories to that of their abelian envelopes. Note that quasi-abelian categories form a special class of non-abelian additive categories which contains in particular the category of locally convex topological vector spaces and the category of filtered abelian groups. In the second part, we define what we mean by an elementary quasi-abelian category and show that sheaves with values in such a category can be manipulated almost as easily as sheaves of abelian groups. In particular, we establish that the Poincaré-Verdier duality and the projection formula hold in this context. The third part is devoted to an application of the results obtained to the cases of filtered and topological sheaves.

***Résumé* (Catégories et faisceaux quasi-abéliens). —** Ce mémoire est divisé en trois parties. Dans la première, nous introduisons la notion de catégorie quasi-abélienne et relions l'algèbre homologique de ces catégories à celle de leurs enveloppes abéliennes. Notons que les catégories quasi-abéliennes forment une classe spéciale de catégories additives non-abéliennes qui contient en particulier la catégorie des espaces vectoriels topologiques localement convexes et la catégorie des groupes abéliens filtrés. Dans la seconde partie, nous définissons ce que nous entendons par catégorie quasi-abélienne élémentaire et montrons que les faisceaux à valeurs dans une telle catégorie sont presque aussi aisés à manipuler que les faisceaux de groupes abéliens. En particulier, nous établissons que la dualité de Poincaré-Verdier et la formule de projection sont valides dans ce contexte. La troisième partie est consacrée à une application des résultats obtenus aux cas des faisceaux filtrés et topologiques.

Contents

INTRODUCTION

To solve various problems of algebraic analysis, it would be very useful to have at hand a good cohomological theory of sheaves with values in categories like that of filtered modules or that of locally convex topological vector spaces. The problem to establish such a theory is twofold. A first difficulty comes form the fact that, since these categories are not abelian, the standard methods of homological algebra cannot be applied in the usual way. A second complication comes from the fact that we need to find conditions under which the corresponding cohomological theories of sheaves are well-behaved. This memoir grew out of the efforts of the author to understand how to modify the classical results in order to be able to treat such situations. The first two chapters deal separately with the two parts of the problem and the last one shows how to apply the theory developed to treat the cases of filtered and topological sheaves.

When we want to develop homological algebra for non abelian additive categories, a first approach is to show that the categories at hand may be endowed with structures of exact categories in the sense of D. Quillen **[14]**. Then, using Paragraph 1.3.22 of **[2]**, it is possible to construct the corresponding derived category and to define what is right or left derived functor.

This approach was followed by G. Laumon in **[10]** to obtain interesting results for filtered $\mathcal{D}$-modules. We checked that it would also be possible to treat similarly the case of locally convex topological vector spaces. However, when one works out the details, it appears that a large part of the results does not come from the particular properties of filtered modules or locally convex topological vector spaces but instead come from the fact that the categories considered are exact categories of a very special kind. In fact, they are first examples of what we call quasi-abelian categories.

To provide a firm ground for applications to other situations, we have found it useful to devote Chapter 1 to a detailed study of the properties of these very special exact categories.

In Section 1.1, after a brief clarification of the notions of images, coimages and strict morphisms in additive categories, we give the axioms that such a category has to satisfy to be quasi-abelian. Next we show that a quasi-abelian category has a canonical exact structure. We conclude by giving precise definitions of the various exactness classes of additive functors between quasi-abelian categories. This is necessary since various exactness properties which are equivalent for abelian categories become distinct in the quasi-abelian case.

Section 1.2 is devoted to the construction of the derived category $\mathcal{D}(\mathcal{E})$ of a quasi-abelian category $\mathcal{E}$ and its two canonical t-structures. We introduce the two corresponding hearts $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ and we make a detailed study of the canonical embedding of $\mathcal{E}$ in $\mathcal{LH}(\mathcal{E})$. In particular, we show that the exact structure of $\mathcal{E}$ is induced by that of the abelian category $\mathcal{LH}(\mathcal{E})$ and that the derived category of $\mathcal{LH}(\mathcal{E})$ endowed with its canonical t-structure is equivalent to $\mathcal{D}(\mathcal{E})$ endowed with its left t-structure. Since the two canonical t-structures are exchanged by duality, it is not necessary to state explicitly the corresponding results for $\mathcal{RH}(\mathcal{E})$. Note that the canonical t-structures of $\mathcal{D}(\mathcal{E})$ and the abelian categories $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ cannot be defined for an arbitrary exact category and give first examples of the specifics of quasi-abelian categories. We end this section with a study of functors from a quasi-abelian category $\mathcal{E}$ to an abelian category $\mathcal{A}$ and show that $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ may in some sense be considered as abelian envelopes of $\mathcal{E}$.

In Section 1.3, we study how to derive an additive functor

$$F : \mathcal{E} \to \mathcal{F}$$

of quasi-abelian categories. After adapting the notions of F-projective and F-injective subcategories to our setting, we generalize the usual criterion for F to be left or right derivable. Next, we study various exactness properties of RF and relate them with the appropriate exactness properties of F. After having clarified how much of a functor is determined by its left or right derived functor and defined the relations of left and right equivalence for quasi-abelian functors, we show that, under mild assumptions, we can associate to F a functor

$$G : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F})$$

which has essentially the same left or right derived functor. Loosely speaking, the combination of this result and those of Section 1.2 shows that from the point of view of homological algebra we do not loose any information by replacing the quasi-abelian category $\mathcal{E}$ by the abelian category $\mathcal{LH}(\mathcal{E})$. We conclude this section by generalizing to quasi-abelian categories, the classical results on projective and injective objects. This leads us to make a careful distinction between projective (resp. injective) and strongly projective (resp. injective) objects of $\mathcal{E}$ and study how they are related with projective and injective objects of $\mathcal{LH}(\mathcal{E})$ or $\mathcal{RH}(\mathcal{E})$.

In Section 1.4, we deal with problems related to projective and inductive limits in quasi-abelian categories. First, we treat the case of products and show mainly that a quasi-abelian category $\mathcal{E}$ has exact (resp. strongly exact) products if and only if $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$) has exact products and the canonical functor

$$\mathcal{E} \to \mathcal{LH}(\mathcal{E}) \quad (\text{resp. } \mathcal{E} \to \mathcal{RH}(\mathcal{E}))$$

is product preserving. The case of coproducts is obtained by duality. After a detailed discussion of the properties of categories of projective or inductive systems of $\mathcal{E}$, we give conditions for projective or inductive limits to be computable in $\mathcal{E}$ as in $\mathcal{LH}(\mathcal{E})$. In this part, we have inspired ourselves from some methods of [**4, 9**]. We conclude by considering the special case corresponding to exact filtering inductive limits.

The last section of Chapter 1 is devoted to the special case of closed quasi-abelian categories (i.e. quasi-abelian categories with an internal tensor product, an internal homomorphism functor and a unit object satisfying appropriate axioms). We show mainly that in such a situation the category of modules over an internal ring is still quasi-abelian. Examples of such categories are numerous (e.g. filtered modules over a filtered ring, normed representations of a normed algebra) but the results obtained will also be useful to treat module over internal rings in a more abstract category like the category $\mathcal{W}$ defined in Chapter 3. We conclude by showing how a closed structure on $\mathcal{E}$ may induce, under suitable conditions, a closed structure on $\mathcal{LH}(\mathcal{E})$.

In Chapter 2 we study conditions on a quasi-abelian category $\mathcal{E}$ insuring that the category of sheaves with values in $\mathcal{E}$ is almost as easily to manipulate as the category of abelian sheaves.

In Section 2.1, we introduce the notions of quasi-elementary and elementary quasi-abelian categories and show that such categories are very easy to manipulate. First, we study the various natural notions of smallness in quasi-abelian categories. We also discuss strict generating sets which play for quasi-abelian categories the role played usually by the generating sets for abelian categories. This allows us to introduce the definitions of quasi-elementary and elementary categories and to show that if $\mathcal{E}$ is quasi-elementary then $\mathcal{LH}(\mathcal{E})$ is a category of functors with values in the category of abelian groups (this an analog of Freyd's result). We also show that the category of ind-objects of a small quasi-abelian category with enough projective objects is a basic example of an elementary quasi-abelian category. We conclude the section with a few results on closed elementary categories.

In Section 2.2, we show that that the category $\mathcal{S}hv(X;\mathcal{E})$ of sheaves on X with values in an elementary quasi-abelian category $\mathcal{E}$ is well-behaved. It is even endowed with internal operations if the category $\mathcal{E}$ is itself closed. Moreover, we show that

$$\mathcal{LH}(\mathcal{S}hv(X;\mathcal{E})) \approx \mathcal{S}hv(X;\mathcal{LH}(\mathcal{E}))$$

and thanks to the results in the preceding sections, we are reduced to work with sheaves in an elementary *abelian* category. Such sheaves where already studied in [**15**] where it is shown that they have most of the usual properties of abelian sheaves. In Section 2.3, we give further examples of how to extend to these sheaves results which are well known for abelian ones. In particular, we prove Poincaré-Verdier duality in this framework. We also prove that if $\mathcal{E}$ is closed and satisfies some mild assumptions then we can establish an internal projection formula and an internal Poincaré-Verdier duality formula by working almost as in the classical case.

Chapter 3 is devoted to applications to filtered and topological sheaves.

In Section 3.1, we study the category of filtered abelian groups and show that this is a closed elementary quasi-abelian with enough projective and injective objects. Its left abelian envelope $\mathcal{R}$ is identified with the category of graded modules over the graded ring $\mathbb{Z}[T]$ following an idea due to Rees. We also show that the category of separated filtered abelian sheaves is a quasi-elementary quasi-abelian category having $\mathcal{R}$ as its left abelian envelope. Since $\mathcal{R}$ is an elementary abelian category, the cohomological theory of sheaves developed in Chapter 2 may be applied to this category and gives a satisfying theory of filtered sheaves. Since most of the results in this section are easy consequences of the general theory, they are often given without proof.

Note that some of the results in this section where already obtained directly in specific situations by various authors (e.g. Illusie, Laumon, Rees, Saito, etc.). However, to our knowledge, the fact that all the classical cohomological formulas for abelian sheaves extend to filtered abelian sheaves was not yet fully established.

It might be a good idea to read this section in parallel with Chapter 1 as it provides a simple motivating example for the abstract theory developed there.

In Section 3.2, we show first that the category of semi-normed spaces in a closed quasi-abelian category with enough projective and injective objects which has the same left abelian envelope as the category of normed-spaces. Applying the results obtained before, we show that the category of ind-semi-normed spaces is a closed elementary quasi-abelian category and that its left abelian envelope $\mathcal{W}$ is a closed elementary abelian category. We also show that the category of locally convex topological vector spaces may be viewed as a (non full) subcategory of $\mathcal{W}$ and that through this identification, the categories of FN (resp. DFN) spaces appear as full subcategories of $\mathcal{W}$. Since the theory developed in Chapter 2 applies to $\mathcal{W}$, we feel that $\mathcal{W}$-sheaves provides a convenient notion of topological sheaves which is suitable for applications in algebraic analysis. Such applications are in preparation and will appear elsewhere.

Note that, in a private discussion some time ago, C. Houzel, conjectured that a category defined through the formula in Corollary 3.2.22 should be a good candidate to replace the category of locally convex topological vector spaces in problems dealing with sheaves and cohomology. He also suggested the name $\mathcal{W}$ since he expected this category to be related to the category of quotient bornological spaces introduced by Waelbroeck. We hope that the material in this paper will have convinced the reader that his insight was well-founded.

Before concluding this introduction, let us point out that discussions we had with M. Kashiwara on a first sketch of this paper lead him to a direct construction of the derived category of the category of FN (resp. DFN) spaces. These categories were used among other tools in **[8]** to prove very interesting formulas for quasi-equivariant $\mathcal{D}$-modules.

Note also that a study of the category of locally convex topological vector spaces along the lines presented here is being finalized by F. Prosmans. However, in this case the category is not elementary and one cannot treat sheaves with values in it along the lines of Chapter 2.

Throughout the paper, we assume the reader has a good knowledge of the theory of categories and of the homological algebra of abelian categories as exposed in standard reference works (e.g. **[11, 12, 16]** and **[3, 5, 7, 17]**). If someone would like an autonomous presentation of the basic facts concerning homological algebra of quasi-abelian categories, he may refer to **[13]** which was based on a preliminary version of Chapter 1.