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DIMERS AND CLUSTER INTEGRABLE SYSTEMS

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ABSTRACT. – We show that the dimer model on a bipartite graph Γ on a torus gives rise to a quantum integrable system of special type, which we call a *cluster integrable system*. The phase space of the classical system contains, as an open dense subset, the moduli space $\mathcal{L}_\Gamma$ of line bundles with connections on the graph Γ. The sum of Hamiltonians is essentially the partition function of the dimer model.

We say that two such graphs Γ_1 and Γ_2 are *equivalent* if the Newton polygons of the corresponding partition functions coincide up to translation. We define elementary transformations of bipartite surface graphs, and show that two equivalent minimal bipartite graphs are related by a sequence of elementary transformations. For each elementary transformation we define a birational Poisson isomorphism $\mathcal{L}_{\Gamma_1} \to \mathcal{L}_{\Gamma_2}$ providing an equivalence of the integrable systems. We show that it is a cluster Poisson transformation, as defined in [10].

We show that for any convex integral polygon N there is a non-empty finite set of minimal graphs Γ for which N is the Newton polygon of the partition function related to Γ. Gluing the varieties $\mathcal{L}_\Gamma$ for graphs Γ related by elementary transformations via the corresponding cluster Poisson transformations, we get a Poisson space $\mathcal{X}_N$. It is a natural phase space for the integrable system. The Hamiltonians are functions on $\mathcal{X}_N$, parametrized by the interior points of the Newton polygon N. We construct Casimir functions whose level sets are the symplectic leaves of $\mathcal{X}_N$.

The space $\mathcal{X}_N$ has a structure of a cluster Poisson variety. Therefore the algebra of regular functions on $\mathcal{X}_N$ has a non-commutative q-deformation to a $*$-algebra $\mathcal{O}_q(\mathcal{X}_N)$. We show that the Hamiltonians give rise to a commuting family of quantum Hamiltonians. Together with the quantum Casimirs, they provide a quantum integrable system. Applying the general quantization scheme [11], we get a $*$-representation of the $*$-algebra $\mathcal{O}_q(\mathcal{X}_N)$ in a Hilbert space. The quantum Hamiltonians act by commuting unbounded selfadjoint operators.

For square grid bipartite graphs on a torus we get *discrete quantum integrable systems*, where the evolution is a cluster automorphism of the $*$-algebra $\mathcal{O}_q(\mathcal{X}_N)$ commuting with the quantum Hamiltonians. We show that the *octahedral recurrence*, closely related to *Hirota's bilinear difference equation* [20], appears this way.

Any graph G on a torus $\mathbb{T}$ gives rise to a bipartite graph Γ_G on $\mathbb{T}$. We show that the phase space $\mathcal{X}$ related to the graph Γ_G has a Lagrangian subvariety $\mathcal{R}$, defined in each coordinate system by a system of monomial equations. We identify it with the space parametrizing resistor networks on G. The pair $(\mathcal{X}, \mathcal{R})$ has a large group of cluster automorphisms. In particular, for a hexagonal grid graph we get

a discrete quantum integrable system on $\mathcal{X}$ whose restriction to $\mathcal{R}$ is essentially given by the *cube recurrence* [33], [4].

The set of positive real points $\mathcal{X}_N(\mathbb{R}_{>0})$ of the phase space is well defined. It is isomorphic to the moduli space of simple Harnack curves with divisors studied in [26]. The Liouville tori of the real integrable system are given by the product of ovals of the simple Harnack curves.

In the sequel [17] to this paper we show that the set of complex points $\mathcal{X}_N(\mathbb{C})$ of the phase space is birationally isomorphic to a finite cover of the Beauville complex algebraic integrable system related to the toric surface assigned to the polygon N.

RÉSUMÉ. – Au modèle des dimères sur un graphe biparti sur le tore, on associe un système intégrable quantique, qu'on appelle *système intégrable de type cluster*.

L'espace des phases classique contient, comme ouvert dense, l'espace des modules $\mathcal{L}_\Gamma$ des fibrés en lignes avec connexion sur le graphe Γ. La somme des hamiltoniens est essentiellement la fonction de partition du modèle des dimères.

Disons que deux graphes Γ_1 et Γ_2 sont *équivalents* si les polygones de Newton des fonctions de partitions correspondantes coïncident à translation près. Nous définissons des transformations élémentaires des graphes bipartis sur une surface, et montrons que deux graphes minimaux et équivalents sont reliés par une suite de transformations élémentaires. Pour chaque transformation élémentaire, nous définissons un isomorphisme de Poisson birationnel $\mathcal{L}_{\Gamma_1} \to \mathcal{L}_{\Gamma_2}$ donnant une équivalence des systèmes intégrables. Nous montrons que c'est une transformation de Poisson de type cluster, comme défini dans [10].

Nous montrons que, pour chaque polygone convexe entier N, il y a un ensemble fini et non-vide de graphes minimaux Γ pour lesquels N est le polygone de Newton de la fonction de partition sous-jacente. Recollant les variétés $\mathcal{L}_\Gamma$ pour les graphes Γ reliés par des transformations élémentaires via les transformations de Poisson correspondantes, on construit un espace de Poisson $\mathcal{X}_N$. C'est un espace de phases naturel pour le système intégrable.

Les hamiltoniens sont des fonctions sur $\mathcal{X}_N$, paramétrées par les points intérieurs de N. On construit des fonctions de Casimir dont les courbes de niveaux sont les feuilles symplectiques de $\mathcal{X}_N$.

L'espace $\mathcal{X}_N$ a une structure de variété de Poisson de type cluster. Alors l'algèbre des fonctions régulières sur $\mathcal{X}_N$ a une q-déformation non-commutative à une $*$-algèbre $\mathcal{O}_q(\mathcal{X}_N)$. Nous montrons que les hamiltoniens fournissent une famille commutative d'hamiltoniens quantiques. Avec les Casimirs quantiques ils engendrent un système intégrable quantique. La méthode générale de [11] donne une $*$-représentation de la $*$-algèbre $\mathcal{O}_q(\mathcal{X}_N)$ dans un espace de Hilbert. Les hamiltoniens quantiques agissent par operateurs auto-adjoints qui commutent entre eux.

Pour le cas d'un quotient de $\mathbb{Z}^2$ sur un tore, nous avons aussi un *système intégrable quantique discret*, dont l'évolution est un automorphisme de type cluster de la $*$-algèbre $\mathcal{O}_q(\mathcal{X}_N)$ commutant avec les hamiltoniens quantiques. Nous montrons que la *récurrence octaédrale* (récurrence de Hirota) apparaît de cette manière.

À n'importe quel graphe G sur le tore on associe un graphe biparti Γ_G sur $\mathbb{T}$. Nous montrons que l'espace des phases $\mathcal{X}$ associé à Γ_G a une sous-variété lagrangienne $\mathcal{R}$, définie dans chaque système de coordonnées par des équations monomiales.

On l'identifie avec l'espace paramétrisant les réseaux de résistances sur G. L'ensemble $(\mathcal{X}, \mathcal{R})$ a un grand groupe d'automorphismes de type cluster. En particulier, pour le graphe hexagonal, on trouve un système intégrable quantique discret sur $\mathcal{X}$ dont la restriction à $\mathcal{R}$ donne la *récurrence cubique* [33, 4].

L'ensemble des points positifs réels $\mathcal{X}_N(\mathbb{R}_{>0})$ de l'espace des phases est bien défini. Il est isomorphe à l'espace de modules des courbes simples de Harnack avec diviseurs étudié dans [26]. Les tores de Liouville du système intégrable réel sont donnés par des produits d'ovales des courbes simples de Harnack.

Dans la suite [17] de cet article, nous montrons que l'ensemble des points complexes $\mathcal{X}_N(\mathbb{C})$ de l'espace des phases est birationnellement isomorphe à un revêtement du système de Beauville relié à la surface torique associée au polygone N.

1. Introduction

A *line bundle* V on a graph Γ is given by assigning a 1-dimensional complex vector space V_v to each vertex of Γ. A *connection* on V is a choice of isomorphism $\phi_{vv'} : V_v \to V_{v'}$ whenever v, v' are adjacent, satisfying $\phi_{v'v} = \phi_{vv'}^{-1}$.

We construct a Poisson structure on the space of line bundles with connections on a bipartite graph embedded on a surface. When the graph is embedded on a torus $\mathbb{T}$ we construct an algebraic integrable system, with commuting Hamiltonians which can be written as Laurent polynomials in natural coordinates on $\mathcal{L}_\Gamma$, the moduli space of line bundles with connections on Γ. The Hamiltonians are sums of dimer covers of Γ.

A *dimer cover* of a graph Γ is a set of edges with the property that every vertex is the endpoint of a unique edge in the cover. Probability measures on dimer covers have been the subject of much recent work in statistical mechanics. In this paper we study some non-probabilistic applications of the dimer model, in particular to the construction of cluster integrable systems.

Below we introduce the key non-technical definitions which we use throughout the paper, and discuss the main results.

1.1. Dimer models and Poisson geometry

A *surface graph* Γ is a graph embedded on a compact oriented surface S whose *faces*, i.e., the connected components of $S - \Gamma$, are contractible. It is homotopy equivalent to an open surface $S_0 \subset S$, obtained by putting a puncture in every face. A *bipartite* graph is a graph with vertices of two types, black and white, such that each edge has one black and one white vertex.

1.1.1. *Conjugate surface graph and Poisson structure*. – Let $\mathcal{L}_\Gamma$ be the moduli space of line bundles with connections on a graph Γ. Any oriented loop L on Γ gives rise to a function W_L on $\mathcal{L}_\Gamma$ given by the monodromy of a line bundle with connection along L. One has $W_{-L} = W_L^{-1}$, where $-L$ is the loop L with the opposite orientation. The monodromies around the loops provide an isomorphism

$$\mathcal{L}_\Gamma \cong \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{C}^*) = H^1(\Gamma, \mathbb{C}^*). \tag{1}$$

For any surface graph Γ, the moduli space $\mathcal{L}_\Gamma$ has the *standard Poisson structure*. Namely, since Γ is homotopy equivalent to the surface S_0, we have $\mathcal{L}_\Gamma = H^1(S_0, \mathbb{C}^*)$. This latter space has a Poisson structure provided by the intersection pairing on $H_1(S_0, \mathbb{Z})$.

Now let Γ be a bipartite surface graph. Then we introduce a *new* Poisson structure on $\mathcal{L}_\Gamma$ related to the standard one by a "twist."

A surface graph is the same thing as a *ribbon graph*: a ribbon graph is a graph with an additional structure given by, for each vertex, a cyclic order of the edges at that vertex. Given a ribbon graph, we can replace its edges by ribbons, getting an oriented surface.

A bipartite surface graph Γ gives rise to a new ribbon graph $\widehat{\Gamma}_w = \widehat{\Gamma}$, the *conjugate graph*, obtained by reversing the cyclic orders at all white vertices[(1)]. The topological surface with boundary corresponding to the ribbon graph $\widehat{\Gamma}_w$ is called the *conjugated surface* $\widehat{S}_w$.

Evidently a line bundle with connection on the graph $\widehat{\Gamma}$ is the same thing as a line bundle with connection on the original graph Γ. So there is a canonical isomorphism

$$\mathcal{L}_\Gamma = \mathcal{L}_{\widehat{\Gamma}}. \tag{2}$$

The standard Poisson structure on the moduli space $\mathcal{L}_{\widehat{\Gamma}}$ combined with the isomorphism (2) provides the Poisson structure on $\mathcal{L}_\Gamma$ we need. Precisely, consider the intersection pairing on $\widehat{S}_w$:

$$\varepsilon_{\widehat{S}_w} : H_1(\widehat{S}_w, \mathbb{Z}) \wedge H_1(\widehat{S}_w, \mathbb{Z}) \longrightarrow \mathbb{Z}. \tag{3}$$

Given two loops L_1, L_2 on Γ, we define the Poisson bracket $\{W_{L_1}, W_{L_2}\}$ by setting

$$\{W_{L_1}, W_{L_2}\} = \varepsilon_{\widehat{S}_w}(L_1, L_2) W_{L_1} W_{L_2} \tag{4}$$

and extending it via the Leibniz rule to a Poisson bracket on the algebra generated by the functions W_L, which coincides with the algebra of regular functions on the variety $\mathcal{L}_\Gamma$.

1.1.2. *Natural coordinates on* $\mathcal{L}_\Gamma$. – The orientation of the surface S induces orientations of the faces of Γ. The group $H_1(\Gamma, \mathbb{Z})$ is generated by the oriented boundaries $\partial(F)$ of the faces F of Γ, whose orientation is induced by the orientation of F, and loops generating $H_1(S, \mathbb{Z})$. The only relation is that the sum of boundaries of all faces is zero. There is an exact sequence, where the map π is induced by the embedding $\Gamma \hookrightarrow S$:

$$0 \longrightarrow \mathrm{Ker}\, \pi \longrightarrow H_1(\Gamma; \mathbb{Z}) \xrightarrow{\pi} H_1(S; \mathbb{Z}) \longrightarrow 0.$$

The subgroup $\mathrm{Ker}\, \pi$ is generated by the oriented boundaries $\partial(F)$ of faces F.

Summarizing, the monodromies of connections around all but one oriented faces F of Γ, augmented by the monodromies around loops generating $H_1(S, \mathbb{Z})$, provide a coordinate system $\{X_i\}$ on the moduli space $\mathcal{L}_\Gamma$. The Poisson structure in these coordinates has a standard quadratic form

$$\{X_i, X_j\} = \varepsilon_{ij} X_i X_j, \qquad \varepsilon_{ij} \in \mathbb{Z}, \tag{5}$$

where ε is the intersection pairing on $\hat{S}_w$.

1.1.3. *Zig-zag loops and the center of the Poisson algebra of functions on* $\mathcal{L}_\Gamma$. – A *zig-zag path* (see [24, 35]) on a ribbon graph Γ is an oriented path on Γ which turns maximally left at white vertices and turns maximally left at black vertices; see Figure 1. Zig-zag paths necessarily close up to form zig-zag loops. There is a bijection

$$\{\text{zig-zag loops on } \Gamma\} \leftrightarrow \{\text{boundaries of the holes on } \widehat{S}_w\}. \tag{6}$$

The orientations of zig-zag loops match the orientations of the boundary loops on $\widehat{S}_w$ induced by the orientation of $\widehat{S}_w$.

(1) Reversing the cyclic order at the black vertices we get a ribbon graph $\widehat{\Gamma}_b$, which differs from $\widehat{\Gamma}_w$ by the orientation.