ISSN 0012-9593

quatrième série - tome 48 fascicule 6 novembre-décembre 2015

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Julien BARRAL

Inverse problems in multifractal analysis of measures

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 48, 2015, p. 1457 à 1510

INVERSE PROBLEMS IN MULTIFRACTAL ANALYSIS OF MEASURES

BY JULIEN BARRAL

ABSTRACT. – Multifractal formalism is designed to describe the distribution at small scales of the elements of $\mathcal{M}_c^+(\mathbb{R}^d)$, the set of positive, finite and compactly supported Borel measures on $\mathbb{R}^d$. It is valid for such a measure μ when its Hausdorff spectrum is the upper semi-continuous function given by the concave Legendre-Fenchel transform of the free energy function τ_μ associated with μ; this is the case for fundamental classes of exactly dimensional measures.

For any function τ candidate to be the free energy function of some $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, we construct such a measure, exactly dimensional, and obeying the multifractal formalism. This result is extended to a refined formalism considering jointly Hausdorff and packing spectra. Also, for any upper semi-continuous function candidate to be the lower Hausdorff spectrum of some exactly dimensional $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$, we construct such a measure.

RÉSUMÉ. – Le formalisme multifractal est un cadre adapté pour décrire la distribution aux petites échelles des mesures de Borel finies positives à support compact dans $\mathbb{R}^d$, dont l'ensemble est ici noté $\mathcal{M}_c^+(\mathbb{R}^d)$. Il est dit valide pour une mesure μ lorsque son spectre de Hausdorff est la fonction semi-continue supérieurement obtenue comme transformée de Legendre-Fenchel concave de sa fonction d'énergie libre τ_μ; c'est le cas pour certaines classes fondamentales de mesures exactement dimensionnelles.

Pour toute fonction τ candidate à être la fonction d'énergie libre d'un élément μ de $\mathcal{M}_c^+(\mathbb{R}^d)$, nous construisons une telle mesure, exactement dimensionnelle, et validant le formalisme. Ce résultat s'étend à un formalisme plus fin considérant simultanément spectres de Hausdorff et de packing. D'autre part, pour toute fonction semi-continue supérieurement candidate à être le spectre de Hausdorff inférieur d'une mesure exactement dimensionnelle, nous construisons une telle mesure.

1. Introduction and main statements

1.1. Inverse problems in multifractal analysis of measures

Let $\mathcal{M}_c^+(\mathbb{R}^d)$ stand for the set of compactly supported Borel positive and finite measures on $\mathbb{R}^d$ ($d \geq 1$), and for $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$ denote by $\mathrm{supp}(\mu)$ the topological support of μ (i.e.,

The author is grateful to De-Jun Feng and Jacques Peyrière for their valuable comments on this work.

the compact set obtained as the complement of those points x for which $\mu(B(x,r)) = 0$ for some $r > 0$, where $B(x,r)$ stands for the closed ball of radius r centered at x).

The upper and lower box dimensions of a bounded set $E \subset \mathbb{R}^d$ will be denoted $\overline{\dim}_B E$ and $\underline{\dim}_B E$ respectively, and its Hausdorff and packing dimensions will be denoted by $\dim_H E$ and $\dim_P E$ respectively (see [33, 60, 70, 81] for introductions to dimension theory).

Multifractal analysis is a natural framework to finely describe geometrically the heterogeneity in the distribution at small scales of the elements of $\mathscr{M}_c^+(\mathbb{R}^d)$. Specifically, if $\mu \in \mathscr{M}_c^+(\mathbb{R}^d)$, this heterogeneity can be described via the lower and upper local dimensions of μ, namely

$$\underline{d}(\mu, x) = \liminf_{r \to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)} \quad \text{and} \quad \overline{d}(\mu, x) = \limsup_{r \to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)},$$

and the level sets

$$E(\mu, \alpha, \beta) = \Big\{x \in \text{supp}(\mu) : \underline{d}(\mu, x) = \alpha,\ \overline{d}(\mu, x) = \beta\Big\} \quad (\alpha \le \beta \in \mathbb{R} \cup \{\infty\}),$$

which form a partition of $\text{supp}(\mu)$ (notice that $E(\mu, \alpha, \beta) = \varnothing$ whenever $\alpha < 0$). The sets

$$\underline{E}(\mu, \alpha) = \Big\{x \in \text{supp}(\mu) : \underline{d}(\mu, x) = \alpha\Big\}, \quad \overline{E}(\mu, \alpha) = \Big\{x \in \text{supp}(\mu) : \overline{d}(\mu, x) = \alpha\Big\},$$

and

$$E(\mu, \alpha) = \underline{E}(\mu, \alpha) \cap \overline{E}(\mu, \alpha) = E(\mu, \alpha, \alpha) \quad (\alpha \in \mathbb{R} \cup \{\infty\})$$

are also very natural, and the most studied in the literature (although the sets defined above are empty if $\alpha < 0$ because μ is a bounded function of Borel sets, it is convenient to include negative values of α in connection with the using along the paper of the Legendre-Fenchel transform of functions defined on $\mathbb{R}$ or $\mathbb{R} \cup \{\infty\}$).

The *lower Hausdorff spectrum* of μ is the mapping defined as

$$\underline{f}_\mu^H : \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H \underline{E}(\mu, \alpha),$$

with the convention that $\dim_H \varnothing = -\infty$, so that $\underline{f}_\mu^H(\alpha) = -\infty$ if $\alpha < 0$. This spectrum provides a geometric hierarchy between the sets $\underline{E}(\mu, \alpha)$, which partition the support of μ. Here, the lower local dimension is emphasized for it provides at any point the best pointwise Hölder control one can have on the measure μ at small scales. However, the upper local dimension is of course of interest, and much attention is paid in general to the sets $E(\mu, \alpha)$ of points at which one has an exact local dimension $\underline{d}(\mu, x) = \overline{d}(\mu, x)$, especially when studying ergodic measures in the context of hyperbolic and more generally non uniformly hyperbolic dynamical systems.

The *Hausdorff spectrum* of μ is the mapping defined as

$$f_\mu^H : \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \dim_H E(\mu, \alpha).$$

Inspired by the observations made by physicists of turbulence and statistical mechanics [42, 40, 41], mathematicians derived, and in many situations justified the heuristic claiming that for a measure possessing a self-conformal like property, its Hausdorff spectrum should be obtained as the Legendre transform of a kind of free energy function, called L^q-spectrum. This gave birth to an abundant literature on the so-called multifractal formalisms [33, 21, 19, 63, 70, 52, 17, 71, 56], which aim at linking the asymptotic statistical properties of a given measure with its fine geometric properties.

To be more specific we need some definitions. Given $I \in \{\mathbb{R}, \mathbb{R} \cup \{\infty\}\}$ and a fonction $f : I \to \mathbb{R} \cup \{-\infty\}$, the domain of f is defined as $\mathrm{dom}(f) = \{x \in I : f(x) > -\infty\}$.

Let $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. If $\mathrm{dom}(\tau) \neq \varnothing$, the concave Legendre-Fenchel transform, or concave conjugate function, of τ is the upper-semi continuous concave function defined as $\tau^* : \alpha \in \mathbb{R} \mapsto \inf\{\alpha q - \tau(q) : q \in \mathrm{dom}(\tau)\}$ (see [77]). We will need a slight extension of this definition.

If $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, $\mathrm{dom}(\tau) \neq \varnothing$, and $0 \in \mathrm{dom}(\tau)$, we define its (extended) concave Legendre-Fenchel transform as

$$\tau^* : \alpha \in \mathbb{R} \cup \{\infty\} \mapsto \begin{cases} \inf\{\alpha q - \tau(q) : q \in \mathrm{dom}(\tau)\} & \text{if } \alpha \in \mathbb{R}, \\ \inf\{\alpha q - \tau(q) : q \in \mathrm{dom}(\tau) \cap \mathbb{R}_-\} & \text{if } \alpha = \infty, \end{cases}$$

with the conventions $\infty \times q = -\infty$ if $q < 0$ and $\infty \times 0 = 0$. Consequently, $\infty \in \mathrm{dom}(\tau^*)$ if and only if $0 = \min(\mathrm{dom}(\tau))$, and in this case $\tau^*(\infty) = -\tau(0) = \max(\tau^*)$. In any case, τ^* is upper semi-continuous over $\mathrm{dom}(\tau^*)$, and concave over the interval $\mathrm{dom}(\tau^*) \setminus \{\infty\}$ (here the notion of upper semi-continuous function is relative to $\mathbb{R} \cup \{\infty\}$ endowed with the topology generated by the open subsets of $\mathbb{R}$ and the sets $(\alpha, \infty) \cup \{\infty\}$, $\alpha \in \mathbb{R}$).

Now, define the (lower) *L^q-spectrum* of $\mu \in \mathcal{M}_c^+(\mathbb{R}^d)$ as

$$\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{r \to 0^+} \frac{\log \sup \left\{ \sum_i \mu(B(x_i, r))^q \right\}}{\log(r)},$$

where the supremum is taken over all the centered packings of $\mathrm{supp}(\mu)$ by closed balls of radius r.

By construction, τ_μ is concave and non decreasing, and

$$-d \leq \tau_\mu(0) = -\overline{\dim}_B \,\mathrm{supp}(\mu) \leq 0 = \tau_\mu(1),$$

so that one always has $\mathbb{R}_+ \subset \mathrm{dom}(\tau_\mu)$; also τ_μ^* takes values in $[0, d] \cup \{-\infty\}$, and $\mathrm{dom}(\tau_\mu^*)$ is a closed subinterval of $\mathbb{R}_+ \cup \{\infty\}$ (see Propositions 1.1 and 1.2).

For $\alpha \in \mathbb{R}$ we always have (see [63, Section 2.7] or [52, Section 3])

$$f_\mu^H(\alpha) \leq \underline{f}_\mu^H(\alpha) \leq \tau_\mu^*(\alpha) \leq \max(\alpha, -\tau_\mu(0)) \leq \max(\alpha, d); \tag{1.1}$$

we also have

$$f_\mu^H(\infty) \leq \tau_\mu^*(\infty),$$

a dimension equal to $-\infty$ meaning that the set is empty (the second inequality is not standard, and will be proved in Section 5; the inequality $\tau_\mu^*(\alpha) \leq \max(\alpha, -\tau_\mu(0))$ is a direct consequence of the definition of τ_μ^* and the fact that $\tau_\mu(1) = 0$).

We notice that due to (1.1), if $f_\mu^H(\alpha) \geq \alpha$ at some α, then $0 \leq \alpha \leq d$ and $f_\mu^H(\alpha) = \tau_\mu^*(\alpha) = \alpha$, so that α is a fixed point of τ_μ^*. Moreover, since $\tau_\mu(1) = 0$ and τ_μ is concave, the set of fixed points of τ_μ^* is the interval $[\tau_\mu'(1^+), \tau_\mu'(1^-)]$.

We will say that μ obeys the multifractal formalism at $\alpha \in \mathbb{R} \cup \{\infty\}$ if $\underline{f}_\mu^H(\alpha) = \tau_\mu^*(\alpha)$, and that the multifractal formalism holds (globally) for μ if it holds at any $\alpha \in \mathbb{R} \cup \{\infty\}$.

If $\underline{f}_\mu^H(\alpha)$ can be replaced by $f_\mu^H(\alpha)$ in the previous definition, we will say that the multifractal formalism holds strongly, and it is in this form that this formalism has been introduced and studied the most. It turns out that in this case one has

$$\dim_H E(\mu, \alpha) = \dim_P E(\mu, \alpha) = \dim_H \underline{E}(\mu, \alpha) = \dim_H \overline{E}(\mu, \alpha) = \tau_\mu^*(\alpha).$$