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MONODROMY AND THE LEFSCHETZ FIXED POINT FORMULA

BY EHUD HRUSHOVSKI AND FRANÇOIS LOESER

To Jan Denef as a token of admiration and friendship on the occasion of his 60th birthday

ABSTRACT. – We give a new proof—not using resolution of singularities—of a formula of Denef and the second author expressing the Lefschetz number of iterates of the monodromy of a function on a smooth complex algebraic variety in terms of the Euler characteristic of a space of truncated arcs. Our proof uses ℓ-adic cohomology of non-archimedean spaces, motivic integration and the Lefschetz fixed point formula for finite order automorphisms. We also consider a generalization due to Nicaise and Sebag and at the end of the paper we discuss connections with the motivic Serre invariant and the motivic Milnor fiber.

RÉSUMÉ. – Nous donnons une nouvelle preuve — n'utilisant pas la résolution des singularités — d'une formule de Denef et du second auteur exprimant le nombre de Lefschetz des itérés de la monodromie d'une fonction sur une variété algébrique complexe en fonction de la caractéristique d'Euler d'un espace d'arcs tronqués. Notre preuve utilise la cohomologie ℓ-adique des espaces non-archimédiens, l'intégration motivique, ainsi que la formule des points fixes de Lefschetz pour les automorphismes d'ordre fini. Nous considérons également une généralisation due à Nicaise et Sebag et la fin de l'article est consacrée aux relations avec l'invariant de Serre motivique et la fibre de Milnor motivique.

1. Introduction

1.1. – Let X be a smooth complex algebraic variety of dimension d and let $f\colon X \to \mathbb{A}^1_{\mathbb{C}}$ be a non-constant morphism to the affine line. Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix a distance function δ on an open neighborhood of x induced from a local embedding of this neighborhood in some complex affine space. For $\varepsilon > 0$ small enough, one may consider the corresponding closed ball $B(x, \varepsilon)$ of radius ε around x. For $\eta > 0$ we denote by D_η the closed disk of radius η around the origin in $\mathbb{C}$.

By Milnor's local fibration Theorem (see [30], [14]), there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exists $0 < \eta < \varepsilon$ such that the morphism f restricts to a fibration, called the Milnor fibration,

$$B(x,\varepsilon) \cap f^{-1}(D_\eta \setminus \{0\}) \longrightarrow D_\eta \setminus \{0\}. \tag{1.1.1}$$

The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x,\varepsilon), \tag{1.1.2}$$

has a diffeomorphism type that does not depend on δ, η and ε. The characteristic mapping of the fibration induces on F_x an automorphism which is defined up to homotopy, the monodromy M_x. In particular the singular cohomology groups $H^i(F_x, \mathbb{Q})$ are endowed with an automorphism M_x, and for any integer m one can consider the Lefschetz numbers

$$\Lambda(M_x^m) = \mathrm{tr}(M_x^m; H^\bullet(F_x, \mathbb{Q})) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(M_x^m; H^i(F_x, \mathbb{Q})). \tag{1.1.3}$$

In [1], A'Campo proved that if x is a singular point of $f^{-1}(0)$, then $\Lambda(M_x^1) = 0$ and this was later generalized by Deligne to the statement that $\Lambda(M_x^m) = 0$ for $0 < m < \mu$, with μ the multiplicity of f at x, cf. [2].

In [13], Denef and Loeser proved that $\Lambda(M_x^m)$ can be expressed in terms of Euler characteristics of arc spaces as follows. For any integer $m \geq 0$, let $\mathcal{L}_m(X)$ denote the space of arcs modulo t^{m+1} on X: a $\mathbb{C}$-rational point of $\mathcal{L}_m(X)$ corresponds to a $\mathbb{C}[t]/t^{m+1}$-rational point of X, cf. [10]. Consider the locally closed subset $\mathcal{X}_{m,x}$ of $\mathcal{L}_m(X)$

$$\mathcal{X}_{m,x} = \{\varphi \in \mathcal{L}_m(X); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}. \tag{1.1.4}$$

Note that $\mathcal{X}_{m,x}$ can be viewed in a natural way as the set of closed points of a complex algebraic variety.

THEOREM 1.1.1 ([13]). – *For every $m \geq 1$,*

$$\chi_c(\mathcal{X}_{m,x}) = \Lambda(M_x^m). \tag{1.1.5}$$

Here χ_c denotes the usual Euler characteristic with compact supports. Note that one recovers Deligne's statement as a corollary since $\mathcal{X}_{m,x}$ is empty for $0 < m < \mu$. The original proof in [13] proceeds as follows. One computes explicitly both sides of (1.1.5) on an embedded resolution of the hypersurface defined by $f = 0$ and checks that both quantities are equal. The computation of the left-hand side relies on the change of variable formula for motivic integration in [10] and the one on the right-hand side on A'Campo's formula in [2]. The problem of finding a geometric proof of Theorem 1.1.1 not using resolution of singularities is raised in [27]. The aim of this paper is to present such a proof.

1.2. – Our approach uses étale cohomology of non-archimedean spaces and motivic integration. Nicaise and Sebag introduced in [33] the analytic Milnor fiber $\mathscr{F}_x$ of the function f at a point x which is a rigid analytic space over $\mathbb{C}((t))$. Let $\mathscr{F}_x^{\mathrm{an}}$ denote its analytification in the sense of Berkovich. Using a comparison theorem of Berkovich, they show that, for every $i \geq 0$, the étale ℓ-adic cohomology group $H^i(\mathscr{F}_x^{\mathrm{an}} \widehat{\otimes} \widehat{\mathbb{C}((t))^{\mathrm{alg}}}, \mathbb{Q}_\ell)$ is isomorphic to $H^i(F_x, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Furthermore, these étale ℓ-adic cohomology groups are naturally endowed with an action of the Galois group $\mathrm{Gal}(\mathbb{C}((t))^{\mathrm{alg}}/\mathbb{C}((t)))$ of the algebraic closure of $\mathbb{C}((t))$, and under this isomorphism the action of the topological generator $(t^{1/n} \mapsto \exp(2i\pi/n)t^{1/n})_{n\geq 1}$ of $\hat{\mu}(\mathbb{C}) = \mathrm{Gal}(\mathbb{C}((t))^{\mathrm{alg}}/\mathbb{C}((t)))$ corresponds to the monodromy M_x.

Another fundamental tool in our approach is provided by the theory of motivic integration developed in [20] by Hrushovski and Kazhdan. Their logical setting is that of the theory $\mathrm{ACVF}(0,0)$ of algebraically closed valued fields of equal characteristic zero, with two sorts VF and RV. If L is a field endowed with a valuation $v : L \to \Gamma(L)$, with valuation ring $\mathcal{O}_L$ and maximal ideal $\mathcal{M}_L$, $\mathrm{VF}(L) = L$ and $\mathrm{RV}(L) = L^\times/(1+\mathcal{M}_L)$. Thus $\mathrm{RV}(L)$ can be inserted in an exact sequence

$$1 \to \mathbf{k}^\times(L) \to \mathrm{RV}(L) \to \Gamma(L) \to 0 \tag{1.2.1}$$

with $\mathbf{k}(L)$ the residue field of L. Let us work with $\mathbb{C}((t))$ as a base field. One of the main result of [20] is the construction of an isomorphism

$$\oint : K(\mathrm{VF}) \longrightarrow K(\mathrm{RV}[*])/I_{\mathrm{sp}} \tag{1.2.2}$$

between the Grothendieck ring $K(\mathrm{VF})$ of definable sets in the VF-sort and the quotient of a graded version $K(\mathrm{RV}[*])$ of the Grothendieck ring of definable sets in the RV-sort by an explicit ideal I_{sp}. At the Grothendieck rings level, the extension (1.2.1) is reflected by the fact that $K(\mathrm{RV}[*])$ may be expressed as a tensor product of the graded Grothendieck rings $K(\Gamma[*])$ and $K(\mathrm{RES}[*])$ for a certain sort RES. A precise definition of RES will be given in 2.2, but let us say that variables in the RES sort range not only over the residue field but also over certain torsors over the residue field so that definable sets in the RES sort are twisted versions of constructible sets over the residue field. This reflects the fact that the extension (1.2.1) has no canonical splitting. Furthermore, there is a canonical isomorphism between a quotient $!K(\mathrm{RES})$ of the Grothendieck ring $K(\mathrm{RES})$ and $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck ring of complex algebraic varieties with $\hat{\mu}$-action, as considered in [12] and [27]. Let $[\mathbb{A}^1]$ denote the class of the affine line. In [20] a canonical morphism

$$\mathrm{EU}_\Gamma : K(\mathrm{VF}) \longrightarrow\ !K(\mathrm{RES})/([\mathbb{A}^1]-1) \tag{1.2.3}$$

is constructed. We shall make essential use of that construction, which is recalled in detail in 2.5. It roughly corresponds to applying the o-minimal Euler characteristic to the Γ-part of the product decomposition of the right-hand side of (1.2.2). Denote by $K(\hat{\mu}\text{-Mod})$ the Grothendieck ring of the category of finite dimensional $\mathbb{Q}_\ell$-vector spaces with $\hat{\mu}$-action. There is a canonical morphism $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \to K(\hat{\mu}\text{-Mod})$ induced by taking the alternating sum of cohomology with compact supports from which one derives a morphism

$$\mathrm{eu}_{\text{ét}} : !K(\mathrm{RES})/([\mathbb{A}^1]-1) \longrightarrow K(\hat{\mu}\text{-Mod}). \tag{1.2.4}$$