ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

Francesco BONSANTE & Gabriele MONDELLO & Jean-Marc SCHLENKER

A cyclic extension of the earthquake flow II

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

A CYCLIC EXTENSION
OF THE EARTHQUAKE FLOW II

by Francesco BONSANTE, Gabriele MONDELLO
and Jean-Marc SCHLENKER

Abstract. – The landslide flow, introduced in [5], is a smoother analog of the earthquake flow on Teichmüller space which shares some of its key properties. We show here that further properties of earthquakes apply to landslides. The landslide flow is the Hamiltonian flow of a convex function. The smooth grafting map sgr taking values in Teichmüller space, which is to landslides as grafting is to earthquakes, is proper and surjective with respect to either of its variables. The smooth grafting map SGr taking values in the space of complex projective structures is symplectic (up to a multiplicative constant). The composition of two landslides has a fixed point on Teichmüller space. As a consequence we obtain new results on constant Gauss curvature surfaces in 3-dimensional hyperbolic or AdS manifolds. We also show that the landslide flow has a satisfactory extension to the boundary of Teichmüller space.

Résumé. – Le flot des glissements de terrain, introduit dans [5], est un analogue régulier du flot des tremblements de terre sur l'espace de Teichmüller, qui partage certaines de ses principales propriétés. Nous montrons ici que d'autres propriétés des tremblements de terre s'appliquent aux glissements de terrain. Le flot des glissements de terrain est le flot hamiltonien d'une fonction convexe. L'application de greffage régulière sgr, à valeur dans l'espace de Teichmüller, qui est aux glissements de terrain ce que le greffage est aux tremblements de terre, est propre et surjective par rapport à chacune de ses variables. L'application de greffage régulière SGr, à valeur dans l'espace des structures projectives complexes, est symplectique (à un facteur multiplicatif près). La composition de deux glissements de terrain a un point fixe dans l'espace de Teichmüller. En conséquence, nous obtenons des résultats nouveaux sur les surfaces à courbure de Gauss constantes dans des variétés de dimension 3 hyperboliques ou AdS. Nous montrons aussi que le flot des glissements de terrain a une extension satisfaisante au bord de l'espace de Teichmüller.

1. Introduction and results

In this paper we consider a closed surface S of genus at least 2. We denote by $\mathscr{T}$ the Teichmüller space of S, considered either as the space of hyperbolic structures on S (considered up to isotopy) or as the space of conformal structures on S (also up to isotopy). We denote by $\mathscr{ML}$ the space of measured laminations on S.

1.1. Earthquakes and landslides

Let γ be a simple closed curve on S, with a weight $w > 0$, and let h be a hyperbolic metric on S. The image of h by the (left) earthquake along the weighted curve $w\gamma$ is obtained by realizing γ as a closed geodesic in (S, h), cutting S open along this geodesic, rotating the right-hand side by a length w in the positive direction, and gluing back. This defines a map $\mathcal{E}(\bullet, w\gamma) : \mathscr{T} \to \mathscr{T}$. Thurston [44] proved that this definition extends from weighted curves to measured laminations, so that we obtain a map:

$$\mathcal{E} : \mathscr{T} \times \mathscr{ML} \to \mathscr{T} .$$

This earthquake map has a number of remarkable properties, of which we can single out, at this stage, the following.

(i) For fixed $\lambda \in \mathscr{ML}$, it defines a flow on $\mathscr{T}$: for all $t_1, t_2 \in \mathbb{R}$, $\mathcal{E}(h, (t_1 + t_2)\lambda) = \mathcal{E}(\mathcal{E}(h, t_1\lambda), t_2\lambda)$.
(ii) Thurston's Earthquake Theorem (see [23, Theorem 2],[31]): for any $h, h' \in \mathscr{T}$, there exists a unique $\lambda \in \mathscr{ML}$ such that $\mathcal{E}(h, \lambda) = h'$.
(iii) McMullen's complex earthquakes [30, Theorem 1.1]: for fixed $\lambda \in \mathscr{ML}$ and $h \in \mathscr{T}$, the map $t \mapsto \mathcal{E}(h, -t\lambda)$ extends as a holomorphic map from the upper half-plane to $\mathscr{T}$, and
(iv) $\mathcal{E}(\bullet, (t + is)\lambda) = \mathrm{gr}(\bullet, s\lambda) \circ \mathcal{E}(\bullet, -t\lambda)$, where $\mathrm{gr}(\bullet, s\lambda) : \mathscr{T} \to \mathscr{T}$ is the grafting map.
(v) The grafting map $\mathrm{gr} : \mathscr{T} \times \mathscr{ML} \to \mathscr{T}$ can be written as the composition $\mathrm{gr} = \Pi \circ \mathrm{Gr}$, where $\mathrm{Gr} : \mathscr{T} \times \mathscr{ML} \to \mathscr{CP}$ is also called the grafting map but with values in the space $\mathscr{CP}$ of complex projective structures on S, and $\Pi : \mathscr{CP} \to \mathscr{T}$ is the forgetful map sending a complex projective structure to the underlying complex structure.
(vi) Thurston proved that the map $\mathrm{Gr} : \mathscr{T} \times \mathscr{ML} \to \mathscr{CP}$ is a homeomorphism (see [21] for a proof).

In [8] we introduced the notion of *landslides*, which can be considered as smooth version of earthquakes. The landslide map $\mathscr{L} : S^1 \times \mathscr{T} \times \mathscr{T} \to \mathscr{T} \times \mathscr{T}$ can be defined in different ways, see below. Still in [8] we showed that properties (i)-(vi) above extend from earthquakes to landslides, with the grafting maps gr and Gr replaced by the corresponding smooth grafting maps sgr' and SGr'.

Here we further consider the analog for landslides of the other well-known properties of the earthquake and grafting maps.

(vii) For a fixed measured lamination λ, the earthquake flow $(t, h) \mapsto \mathcal{E}(h, t\lambda)$ is the Hamiltonian flow of the length function of $\frac{1}{2}\lambda$, considered as a function on $\mathscr{T}$, with respect to the Weil-Petersson symplectic structure (see [44], [52] and [23]).
(viii) The length of a measured lamination is a convex function on $\mathscr{T}$ with respect to the Weil-Petersson metric, as proved by Wolpert [53] (see also [51]).

(ix) Given two measured laminations $\lambda, \mu \in \mathcal{ML}$ which fill S, the composition $\mathcal{E}(\bullet, \lambda) \circ \mathcal{E}(\bullet, \mu) : \mathcal{T} \to \mathcal{T}$ has a fixed point (conjectured to be unique), see [9, Theorem 1.1].

(x) For fixed $\lambda \in \mathcal{ML}$, the map $\mathrm{gr}(\bullet, \lambda) : \mathcal{T} \to \mathcal{T}$ is a homeomorphism (see [36, Theorem A]), and, for $h \in \mathcal{T}$ fixed, the map $\mathrm{gr}(h, \bullet) : \mathcal{ML} \to \mathcal{T}$ is a homeomorphism (see [12, Theorem 1.1]).

(xi) The cotangent space $T^*\mathcal{T}$ can be identified with the product $\mathcal{T} \times \mathcal{ML}$ through the map $d\ell : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$ which sends (h, λ) to the differential at h of $d\ell_\lambda$--in particular this map is one-to-one (see [27, Lemma 2.3]).

(xii) The grafting map $\mathrm{Gr} : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ can be composed with $(d\ell)^{-1} : T^*\mathcal{T} \to \mathcal{T} \times \mathcal{ML}$ to obtain a map from $T^*\mathcal{T}$ to $\mathcal{CP}$. By [27, Lemma 1.1 and Theorem 1.2] this map is actually C^1 (although $\mathcal{ML}$ does not have a natural C^1-structure) and symplectic (up to a constant factor), when one considers on $\mathcal{CP}$ the real symplectic structure equal to the real part of the Goldman symplectic structure on the space of representations of $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{C})$.

We will prove that those properties extend to landslides, except for point (x) for which we only prove here that the corresponding maps in the landslide setting are onto. (We also believe that those maps are one-to-one, but could not prove it.)

We will see that points (ix) and (x) can be translated in terms of 3-dimensional hyperbolic or anti-de Sitter geometry.

In addition we will show (see Section 1.7 for a more precise statement) that

(xiii) the landslide map has a satisfactory extension to the space $\mathcal{FML} \subset \mathcal{ML} \times \mathcal{ML}$ of projective classes of filling pairs of measured laminations on S, whose projectivization can be considered as a bordification of $\mathcal{T} \times \mathcal{T}$; this extension is Hamiltonian for the symplectic structure equal to the sum of the Thurston symplectic forms on the two factors of $\mathcal{ML} \times \mathcal{ML}$.

1.2. The landslide flow is Hamiltonian

We will first define a function F on $\mathcal{T} \times \mathcal{T}$ that plays for landslides the role that the length of a measured lamination plays for earthquakes. Recall that given two hyperbolic metrics h and $h^\star$ on S, there is a unique minimal Lagrangian map m isotopic to the identity from (S, h) to $(S, h^\star)$ (see [29] and [39, Proposition 2.12]). This map can be characterized by the existence of a bundle morphism $b : TS \to TS$ which has determinant 1, is self-adjoint for h and satisfies the Codazzi equation $d^\nabla b = 0$, and such that $m^* h^\star(\bullet, \bullet) = h(b\bullet, b\bullet)$. We call b the *Labourie operator* of the pair $(h, h^\star)$ and c the *center* of $(h, h^\star)$, namely the conformal structure (up to isotopy) underlying the metric $h + m^* h^\star$.

DEFINITION 1.1. – Let $F : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ be the function defined as

$$F(h, h^\star) = \int_S \mathrm{tr}(b) \, da_h \,,$$

where b is the Labourie operator of the pair $(h, h^\star)$ and da_h is the area element associated to h. Given a fixed $h^\star \in \mathcal{T}$, we will also denote by $F_{h^\star} : \mathcal{T} \to \mathbb{R}$ the function defined as $F_{h^\star}(h) := F(h, h^\star)$.