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FIXED POINTS FOR BOUNDED ORBITS IN HILBERT SPACES

BY MAXIME GHEYSENS AND NICOLAS MONOD

ABSTRACT. – Consider the following property of a topological group G: every continuous affine G-action on a Hilbert space with a bounded orbit has a fixed point. We prove that this property characterizes amenability for locally compact σ-compact groups (e.g., countable groups).

Along the way, we introduce a "moderate" variant of the classical induction of representations and we generalize the Gaboriau-Lyons theorem to prove that any non-amenable locally compact group admits a probabilistic variant of discrete free subgroups. This leads to the "measure-theoretic solution" to the von Neumann problem for locally compact groups.

We illustrate the latter result by giving a partial answer to the Dixmier problem for locally compact groups.

RÉSUMÉ. – Nous considérons la propriété suivante pour un groupe topologique G : toute action affine continue de G sur un espace hilbertien ayant une orbite bornée a un point fixe. Nous montrons qu'elle caractérise la moyennabilité des groupes localement compacts dénombrables à l'infini (en particulier des groupes discrets dénombrables).

Pour ce faire, nous introduisons une variante « modérée » de l'induction des représentations et nous généralisons le théorème de Gaboriau-Lyons pour montrer que tout groupe localement compact non moyennable admet, dans un sens probabiliste, des sous-groupes libres discrets. Ceci fournit une « solution au sens de la mesure » au problème de von Neumann pour les groupes localement compacts.

Nous illustrons ce dernier résultat en fournissant une réponse partielle au problème de Dixmier pour les groupes localement compacts.

1. Introduction

A topological group G is *amenable* if every convex compact G-space $K \neq \emptyset$ has a fixed point. The precise meaning of this definition is that K is a non-empty convex compact subset of a locally convex topological vector space V and that G has a continuous affine action on K. It is equivalent to consider only the case where this is given by a continuous affine (or linear) G-representation on V preserving K (for K can be identified with the state space

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of the unital ordered space of affine continuous functions on K). Moreover, we can assume V and K separable if G is, for instance, locally compact σ-compact.

It is well-known that any such K is isomorphic (i.e., affinely homeomorphic) to a convex compact subspace of a Hilbert space [33, p. 31]. Does it follow that amenability is characterized as a fixed point property for affine actions on Hilbert spaces? after all, preserving a weakly compact set in Hilbert space is equivalent to having a bounded orbit. (The distinction between weak and strong compactness will be further discussed in Section 9.)

The answer is a resounding *no*. First of all, an action on K need not extend to the ambient Hilbert space (see Section 9). Moreover, G-actions on V preserving K sometimes have fixed points outside K only, compare e.g., [3].

In any case, even the statement is wrong! Indeed, there are non-amenable groups with the fixed point property for any continuous affine action on any reflexive Banach space. This holds for instance for the group of all permutations of an infinite countable set, which is non-amenable (as a discrete group). Indeed, Bergman established the strong uncountable cofinality property for this group [4] and the latter implies this fixed point property. Indeed the action is automatically uniformly equicontinuous (as seen from the isometric action on the metric space of equivalent norms) and hence fixes a point by Ryll-Nardzewski. See Prop. 1.30 of [51] for a Polish variant of this fact.

In contrast, we prove that such a characterization does hold for countable groups and more generally locally compact σ-compact groups:

THEOREM A. – *Let G be a locally compact σ-compact group.*

Suppose that every continuous affine G-action on a separable Hilbert space with a bounded orbit has a fixed point. Then G is amenable.

(In view of the fixed point definition of amenability, this yields a necessary and sufficient condition and it follows furthermore that a fixed point can be found in the closed convex hull of any bounded orbit.)

In this setting, we recall that the fixed point property *without* assuming bounded orbits characterizes compact groups by a result of Rosendal [51, Thm. 1.4].

Our proof takes a curious path: we first construct a very specific example of a group without the fixed point property, and then we pull ourselves by our bootstraps until we reach all non-amenable groups. This process is described below; we would be curious to know if there is a direct proof.

In the absence of a direct proof, the scenic route taken to the conclusion leads us to introduce *moderate induction* and to establish the existence of *tychomorphisms* from free groups to non-amenable locally compact groups, after proving a generalization of the theorem of Gaboriau and Lyons to the locally compact setting. This solves the "measurable von Neumann problem" for locally compact groups, see Theorem B below.

REMARK. – Our task is thus to construct fixed point free actions on Hilbert spaces that have a bounded orbit. We point out that such actions always have some *unbounded* orbit too. Otherwise, an application of the Banach-Steinhaus principle would show that the linear part of the action is uniformly bounded in operator norm; this would however produce a fixed point, for instance by taking a circumcenter under an invariant uniformly convex norm [2, Prop. 2.3], or using Ryll-Nardzewski.

Discrete outline of the proof

We shall first explain our proof in the special case of countable groups without any topology. Our first step is to obtain some example, *any example at all*, of a group G with a fixed point free action on a Hilbert space with a bounded orbit.

Let thus μ be a probability measure on G; this amounts to a non-negative function of sum one. Our Hilbert space is $V = \ell^2(G,\mu)/\mathbf{R}$, the quotient of $\ell^2(G,\mu)$ by the subspace of constant functions. We endow V with the linear representation induced by the left translation action of G on $\ell^2(G,\mu)$, which indeed preserves the subspace of constants. For this action to be well-defined and to be continuous we need to impose a condition on how μ behaves under translations. It turns out that such a μ exists for every countable group; it will be constructed as a negative exponential of suitable length functions on G.

To turn this linear representation into an affine action, we need a 1-cocycle $G \to V$. The action is fixed point free and with a bounded orbit if this cocycle is non-trivial in cohomology and bounded. The extension of G-representations

$$0 \longrightarrow \mathbf{R} \longrightarrow \ell^2(G,\mu) \longrightarrow V \longrightarrow 0$$

can be analyzed by standard cohomological arguments and it suffices to show that there is an $\mathbf{R}$-valued 2-cocycle on G which is non-trivial in cohomology and bounded. Such cocycles are known to exists for various groups G, for instance (compact hyperbolic) surface groups. Thus we have a first example.

In order to produce more examples, we want to show that our G-action on V can be "induced" to an H-action on another Hilbert space W whenever H is a group containing G. Classically, W would be a space of maps $H/G \to V$. We shall imitate the first step of our construction by considering ℓ^2-maps with respect to a suitable probability measure on H/G; once again, such a measure will exist as soon as H is countable.

At this point, we have constructed a fixed point free action on a separable Hilbert space with a bounded orbit for any countable group containing a surface group. The same statement holds with surface groups replaced by free groups since fixed point properties trivially pass to quotients. We now reach a fundamental obstacle popularized by the *von Neumann problem*: the class of groups containing a free subgroup is still far from the class of non-amenable countable groups.

However, it was proved by Gaboriau-Lyons [22] that in an ergodic-theoretical sense, any non-amenable discrete group admits free orbits of free groups as subrelations. As explained in [41, § 5], such measure-theoretical analogues of subgroup embeddings, viewed as "randembeddings," are suitable for the induction of representations and of cocycles. Therefore, we can complete the proof of Theorem A *for discrete groups* by generalizing the above induction method from subgroups to randembeddings.