ISSN 0012-9593

quatrième série - tome 51 *fascicule 6* *novembre-décembre 2018*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Benoît CLAUDON & Frank LORAY & Jorge Vitório PEREIRA & Frédéric TOUZET

Compact leaves of codimension one holomorphic foliations on projective manifolds

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 540 €. Hors Europe : 595 € ($ 863). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 51, 2018, p. 1457 à 1506

COMPACT LEAVES OF CODIMENSION ONE HOLOMORPHIC FOLIATIONS ON PROJECTIVE MANIFOLDS

BY Benoît CLAUDON, Frank LORAY, Jorge Vitório PEREIRA AND Frédéric TOUZET

Abstract. – This article studies codimension one foliations on projective manifolds having a compact leaf (free of singularities). It explores the interplay between Ueda theory (order of flatness of the normal bundle) and the holonomy representation (dynamics of the foliation in the transverse direction). We address in particular the following problems: existence of foliation having as a leaf a given hypersurface with topologically torsion normal bundle, global structure of foliations having a compact leaf whose holonomy is abelian (resp. solvable), and factorization results.

Résumé. – Cet article étudie les feuilletages de codimension 1 sur les variétés projectives admettant une feuille compacte (ne rencontrant pas le lieu singulier du feuilletage). Les interactions entre la théorie de Ueda (ordre de platitude du fibré normal de la feuille) et la représentation d'holonomie (dynamique du feuilletage dans la direction transverse) sont explorées. Nous envisageons en particulier les problématiques suivantes : existence de feuilletages admettant pour feuille une hypersurface donnée possédant un fibré normal topologiquement de torsion, étude de la structure globale des feuilletages ayant une feuille compacte d'holonomie abélienne (resp. résoluble) et résultats de factorisations.

1. Introduction

Let X be a complex manifold and $\mathscr{F}$ a codimension one singular holomorphic foliation on it. If Y is a compact leaf of $\mathscr{F}$ (the foliation is regular along Y) then the topology/dynamics of $\mathscr{F}$ near Y is determined by the holonomy of Y. Given a base point $p \in Y$ and a germ of transversal $(\Sigma, p) \simeq (\mathbb{C}, 0)$ to Y at p, we can lift paths on Y to nearby leaves in order to obtain the holonomy representation of $\mathscr{F}$ along Y

$$\rho : \pi_1(Y, p) \longrightarrow \mathrm{Diff}(\mathbb{C}, 0) .$$

This work was initiated during a research stay of the first author at IMPA. He would like to warmly thank the institute for its hospitality and excellent working conditions. The third author is supported by CNPq and FAPERJ, and BC, FL and FT benefit from support of CNRS and ANR-16-CE40-0008 project "Foliage".

 doi:10.24033/asens.2379

The purpose of this article is to investigate which representations as above can occur as the holonomy representation of a compact leaf of a codimension one foliation on a compact Kähler manifold, and how they influence the geometry of the foliation $\mathcal{F}$.

1.1. Previous results

To the best of our knowledge the first work to explicitly study compact leaves of foliations on compact manifolds is due to Sad [38]. One of his main results, [38, Theorem 1], is stated below.

THEOREM 1.1. – *Let $C_0 \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 3$ and P be a set of d^2 very general points on C_0. If S is the blow-up of $\mathbb{P}^2$ along P and C is the strict transform of C_0 in S then C is not a compact leaf of any foliation on S.*

This result answered negatively a question of Demailly, Peternell and Schneider [20, Example 2] about the existence of foliations having a compact leaf on the blow-up of $\mathbb{P}^2$ along 9 very general points. Sad (loc. cit.) obtained other results for the blow-up of smooth cubics which were later subsumed by the following result of Brunella build upon the classification of foliations according to their Kodaira dimension, see [7, Chapter 9, Corollary 2].

THEOREM 1.2. – *Let $\mathcal{F}$ be a foliation on a projective surface S and suppose $\mathcal{F}$ admits an elliptic curve E as a compact leaf. Then, either E is a (multiple) fiber of an elliptic fibration or, up to ramified coverings and birational maps, E is a section of a $\mathbb{P}^1$-bundle. In the former case, $\mathcal{F}$ is the elliptic fibration itself, or is turbulent with respect to it; and in the latter case, $\mathcal{F}$ is a Riccati foliation.*

Bott's partial connection induces a flat connection on the normal bundle of compact leaves of foliations. Hence, our subject is also closely related to the study of smooth divisors with numerically trivial normal bundle on projective or compact Kähler manifolds. Serre constructed examples of curves on ruled surfaces with trivial normal bundle, having Stein complement but without non constant regular (algebraic) function [23, Chapter VI, Example 3.2]. Precisely, given an elliptic curve C, consider the unique unsplit extension $0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$; on the total space $X = \mathbb{P}(E)$, the curve $Y \simeq C$ defined by the embedding $\mathcal{O}_C \to E$ provides us with such an example. Motivated by these examples, Hartshorne asked if the complement of a curve C on a projective surface S is Stein whenever $C^2 \geq 0$ and $S - C$ contains no complete curves [23, Chapter VI, Problem 3.4]. This question was answered negatively by Ogus, who exhibits examples of rational surfaces [35, Section 4] containing elliptic curves with numerically trivial normal bundle, without complete curves in its complement, and which are not Stein. Both Ogus and Serre examples carry global foliations smooth along the curves. A variation on Ogus example is presented in Example 4.5 below.

Ueda carried on the study of smooth curves on surfaces with numerically trivial normal bundle, looking for obstructions to the existence of certain kind of foliations smooth along the curves. The Ueda type $k \in \{1, 2, 3, \ldots, \infty\}$ of Y (utype(Y) for short) is, roughly speaking, defined by the first infinitesimal neighborhood of Y for which the flat unitary foliation of NY does not extend as a foliation with linearizable holonomy. See Section 2.1 for a precise definition. There, we also review some of the results of Ueda and Neeman following [42] and [34].

1.2. Existence

Although most of our results deal with the global setting, where X is assumed to be projective or compact Kähler, we first prove the existence of a formal foliation in the semi-local setting.

THEOREM A. – *Let $Y \subset X$ a smooth curve embedded in a germ of surface X such that $Y^2 = 0$. If one denotes by $Y(\infty)$ the formal completion of X along Y, then Y is a leaf of a (formal) regular foliation $\hat{\mathcal{F}}$ on $Y(\infty)$.*

Turning to the global setting, our second result concerns the existence of foliations smooth along hypersurfaces with topologically torsion (or flat) normal bundle.

THEOREM B. – *Let X be a compact Kähler manifold and Y be a smooth hypersurface in X with normal bundle NY. Assume there exists an integer k and a flat line-bundle $\mathcal{L}$ on X such that $\mathcal{L}_{|Y} = NY^{\otimes k}$. If* $\mathrm{utype}(Y) \geq k$*, then there exists a rational 1-form Ω with coefficients in $\mathcal{L}^*$ and polar divisor equal to $(k+1)Y$. Moreover, if ∇ is the unitary flat connection on $\mathcal{L}^*$, then $\nabla(\Omega) = 0$, and Ω defines a transversely affine foliation on X which admits Y as a compact leaf.*

We refer to [18] for the notion of transversely affine foliation.

When $\mathrm{utype}(Y) > k$, where k is the analytical order of NY, this result is due to Neeman [34, Theorem 5.1, p. 109], where it is proved that Y fits into a fibration.

THEOREM 1.3. – *Let X be a compact Kähler manifold and Y be a smooth hypersurface in X with torsion normal bundle of order k. Assume that the Ueda type of Y satisfies* $\mathrm{utype}(Y) > k$*, then* $\mathrm{utype}(Y) = \infty$ *and kY is a fiber of a fibration on X.*

The reader interested in other fibration existence criteria may consult [21, 36, 40]. Remark that, when NY is analytically trivial, Theorem B implies the existence of global foliations on X for which Y is a compact leaf. This fact is in contrast with Theorem 1.1. Sufficient and necessary conditions for the existence of a global foliation on compact Kähler manifold having a given compact leaf remain elusive. For instance, we are not aware of smooth hypersurfaces with torsion normal bundle and Ueda type strictly smaller than the order of the normal bundle which are not compact leaves. For more on this matter see the discussion at Section 4.4.

1.3. Abelian holonomy

For arbitrary representation $\rho : \pi_1(Y, p) \longrightarrow \mathrm{Diff}(\mathbb{C}, 0)$ there exists a complex manifold U containing Y as a hypersurface, and a foliation $\mathcal{F}$ on U leaving Y invariant and with holonomy representation along Y given by ρ, see [25]. If we start with an abelian representation ρ, it is well-known (see Section 5.4) that there exists a formal meromorphic closed 1-form in $Y(\infty)$, the completion of U along Y, defining $\mathcal{F}_{|Y(\infty)}$. In many cases there do not exist (convergent) meromorphic closed 1-forms defining $\mathcal{F}$ on U, even if we restrict U. Our third result says that, for codimension one foliations on projective manifolds, this can only happen when the holonomy is linearizable.