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A FUNCTIONAL ANALYSIS PROOF OF GROMOV'S POLYNOMIAL GROWTH THEOREM

BY NARUTAKA OZAWA

ABSTRACT. – The celebrated theorem of Gromov asserts that any finitely generated group with polynomial growth contains a nilpotent subgroup of finite index. Alternative proofs have been given by Kleiner and others. In this note, we give yet another proof of Gromov's theorem, along the lines of Shalom and Chifan-Sinclair, which is based on the analysis of reduced cohomology and Shalom's property H_{FD}.

RÉSUMÉ. – Un résultat célèbre de Gromov affirme que tout groupe finiment engendré de croissance polynomiale contient un sous-groupe nilpotent d'indice fini. Des preuves alternatives de ce résultat ont été données par Kleiner, entre autres. Dans cette note, nous donnons une nouvelle preuve du théorème de Gromov, dans l'esprit de résultats de Shalom et Chifan-Sinclair, reposant sur l'analyse de la cohomologie réduite et la propriété H_{FD} de Shalom.

1. Introduction

The celebrated theorem of Gromov ([10, 7]) asserts that any finitely generated group with weakly polynomial growth contains a nilpotent subgroup of finite index. Here a group G is said to have *weakly polynomial growth* if $\liminf \log|S^n|/\log n < \infty$ for any finite generating subset S such that $1 \in S = S^{-1}$. Alternative proofs have been given by Kleiner and others ([13, 18, 12, 3]). In this note, we give yet another proof of Gromov's theorem, along the lines of Shalom ([17]) and Chifan-Sinclair ([5]), which is based on the analysis of reduced cohomology and Shalom's property H_{FD}.

Let $\pi\colon G \curvearrowright \mathcal{H}$ be a unitary representation. Recall that a 1-*cocycle* of G with coefficients in π is a map $b\colon G \to \mathcal{H}$ which satisfies

$$\forall g, x \in G \quad b(gx) = b(g) + \pi_g b(x).$$

A 1-*coboundary* is a 1-cocycle of the form $b(g) = \xi - \pi_g \xi$ for some $\xi \in \mathcal{H}$, and an *approximate* 1-*coboundary* is a 1-cocycle that is a pointwise limit of 1-coboundaries. The

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spaces of 1-cocycles, 1-coboundaries, approximate 1-coboundaries are written respectively by $Z^1(G,\pi)$, $B^1(G,\pi)$, and $\overline{B^1(G,\pi)}$, and so the *reduced cohomology* space $\overline{H^1}(G,\pi)$ is $Z^1(G,\pi)/\overline{B^1(G,\pi)}$. It is proved by Mok and Korevaar-Schoen ([15, 14], see also [16] and Theorem A in Appendix) that any finitely generated group G without Kazhdan's property (T) admits a unitary representation π with $\overline{H^1}(G,\pi) \neq 0$. A group G is said to have *Shalom's property* H_{FD} if $\overline{H^1}(G,\pi) \neq 0$ implies that π is not weakly mixing. Here π is said to be *weakly mixing* if $\mathcal{H}$ admits no nonzero finite-dimensional $\pi(G)$-invariant subspaces. We recall that infinite amenable groups, and in particular groups with weakly polynomial growth, do not have property (T) (see e.g., [4, Chapter 12]). Thus, if such a group has property H_{FD}, then it has a finite-dimensional unitary representation π with $\overline{H^1}(G,\pi) \neq 0$. Shalom has observed that a proof of property H_{FD} for a group with weakly polynomial growth implies Gromov's theorem (see [17, Section 6.7] and [19]). In this paper, we prove that a group with slow entropy growth has property H_{FD}, thus giving a new proof of Gromov's theorem. Here we say G has *slow entropy growth* if there is a non-degenerate finitely-supported symmetric probability measure μ on G with $\mu(e) > 0$ such that

$$\liminf_n n(H(\mu^{*n+1}) - H(\mu^{*n})) < \infty,$$

where H is the entropy functional. This property is formerly weaker than but probably equivalent to weakly polynomial growth (see Section 3).

THEOREM. – *A finitely generated group with slow entropy growth has property* H_{FD}.

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2. Reduced cohomology and harmonic 1-cocycles

Let G be a finitely generated group and fix a non-degenerate finitely-supported symmetric probability measure μ with $\mu(e) > 0$. Let $\pi\colon G \curvearrowright \mathcal{H}$ be a unitary representation. We first recall the fact that every element in the reduced cohomology space $\overline{H^1}(G,\pi)$ is uniquely represented by a μ-harmonic 1-cocycle (see [11, 1]). The space $Z^1(G,\pi)$ of 1-cocycles is a Hilbert space under the norm

$$\|b\|_{Z^1(G,\pi)} := \Big(\sum_x \mu(x)\|b(x)\|^2\Big)^{1/2},$$

and the space $\overline{B^1(G,\pi)}$ agrees with the closure of $B^1(G,\pi)$ in the Hilbert space $Z^1(G,\pi)$. We observe that $b \in Z^1(G,\pi)$ is orthogonal to $B^1(G,\pi)$ if and only if it is μ-*harmonic*: $\sum_x \mu(x)b(x) = 0$ or equivalently $\sum_x \mu(x)b(gx) = b(g)$ for all $g \in G$. Indeed, this follows from the identities $b(x^{-1}) + \pi_x^{-1}b(x) = b(e) = 0$ and

$$\sum_x \mu(x)\langle b(x), \xi - \pi_x\xi\rangle = 2\langle \sum_x \mu(x)b(x), \xi\rangle.$$

Since $Z^1(G,\pi) = \overline{B^1(G,\pi)} \oplus B^1(G,\pi)^\perp$ as a Hilbert space, $\overline{H^1}(G,\pi)$ can be identified with the space $B^1(G,\pi)^\perp$ of μ-harmonic 1-cocycles.

By the above discussion, we may concentrate on μ-harmonic 1-cocycles. For any μ-harmonic 1-cocycle b, one has $\sum_x \mu^{*n}(x)\|b(x)\|^2 = n\sum_x \mu(x)\|b(x)\|^2$ (by induction on n). In this section, we give a better inequality, which is inspired by the work of Chifan and Sinclair ([5]). Let $\mathcal{H} \otimes \bar{\mathcal{H}}$ denote the Hilbert space tensor product of the Hilbert space $\mathcal{H}$ and its complex conjugate $\bar{\mathcal{H}}$. We recall that π is weakly mixing if and only if the unitary representation $\pi \otimes \bar{\pi}$ on $\mathcal{H} \otimes \bar{\mathcal{H}}$ has no nonzero invariant vectors. Indeed, $\mathcal{H} \otimes \bar{\mathcal{H}}$ can be identified with the space $S_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$, and under this identification $\pi_g \otimes \bar{\pi}_g$ becomes the conjugation action $\operatorname{Ad}\pi_g$ of π_g on $S_2(\mathcal{H})$ (see e.g., Section 13.5 in [4]). Since any nonzero Hilbert-Schmidt operator (which is $\operatorname{Ad}\pi_g$-invariant) is compact and has a nonzero finite-dimensional eigenspace (which is π_g-invariant), our claim follows.

LEMMA. – *Let $b\colon G \to \mathcal{H}$ be a μ-harmonic* 1*-cocycle with coefficients in a weakly mixing unitary representation π. Then one has*

$$\frac{1}{n}\Big\|\sum_x \mu^{*n}(x)(b(x)\otimes \bar{b}(x))\Big\|_{\mathcal{H}\otimes\bar{\mathcal{H}}} \to 0.$$

In particular,

$$\sup_{\xi\in\mathcal{H},\,\|\xi\|\le 1} \frac{1}{n}\sum_x \mu^{*n}(x)|\langle b(x),\xi\rangle|^2 \to 0.$$

Proof. – Since b is μ^{*n}-harmonic for every n, one has for every n and $g \in G$

$$\sum_x \mu^{*n}(x)(b(gx)\otimes\bar{b}(gx)) = b(g)\otimes\bar{b}(g) + (\pi_g\otimes\bar{\pi}_g)\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x)).$$

Thus, putting $\zeta := \sum_x \mu(x)(b(x)\otimes\bar{b}(x))$ and $T := \sum_g \mu(g)(\pi_g\otimes\bar{\pi}_g)$, one has

$$\begin{aligned}\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x)) &= \sum_{g,x}\mu(g)\mu^{*n-1}(x)(b(gx)\otimes\bar{b}(gx))\\ &= \zeta + T\sum_x \mu^{*n-1}(x)(b(x)\otimes\bar{b}(x))\\ &= \cdots = (1+T+\cdots+T^{n-1})\zeta.\end{aligned}$$

Since π is weakly mixing, $\pi\otimes\bar{\pi}$ admits no nonzero invariant vectors, and hence by strict convexity of a Hilbert space, 1 is not an eigenvalue of the self-adjoint contraction T. Hence, the measure $m(\cdot) := \langle E_T(\cdot)\zeta,\zeta\rangle$, associated with the spectral resolution E_T of T, is supported on $[-1,1]$ and satisfies $m(\{1\}) = 0$. Thus, one has

$$\frac{1}{n}\Big\|\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x))\Big\|_{\mathcal{H}\otimes\bar{\mathcal{H}}} = \Big(\int_{-1}^{1}\Big|\frac{1+t+\cdots+t^{n-1}}{n}\Big|^2\,dm(t)\Big)^{1/2} \to 0$$

by Bounded Convergence Theorem. The second statement follows from the first, because $|\langle b(x),\xi\rangle|^2 = \langle b(x)\otimes\bar{b}(x), \xi\otimes\bar{\xi}\rangle$. □