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DYNAMICAL COMPLEXITY AND CONTROLLED OPERATOR K-THEORY

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DYNAMICAL COMPLEXITY AND CONTROLLED OPERATOR K-THEORY

by

E. GUENTNER, R. WILLETT & G. YU

Abstract. — In this paper, we introduce a property of topological dynamical systems that we call finite dynamical complexity. For systems with this property, one can in principle compute the K-theory of the associated crossed product C^*-algebra by splitting it up into simpler pieces and using the methods of controlled K-theory. The main part of the paper illustrates this idea by giving a new proof of the Baum-Connes conjecture for actions with finite dynamical complexity.

We have tried to keep the paper as self-contained as possible: we hope the main part will be accessible to someone with the equivalent of a first course in operator K-theory. In particular, we do not assume prior knowledge of controlled K-theory, and use a new and concrete model for the Baum-Connes conjecture with coefficients that requires no bivariant K-theory to set up.

***Résumé.* (Complexité dynamique et K-théorie contrôlée).** — Nous introduisons une nouvelle propriété des systèmes dynamiques topologiques, que nous appelons complexité dynamique finie. Les produits-croisés de C^*-algèbres associés aux systèmes dynamiques ayant cette propriété peuvent être décomposés en parties plus simples, ce qui permet de calculer leurs groupes de K-théorie, via des méthodes de K-théorie contrôlée.

Dans cet article, nous illustrons cette idée en donnant une nouvelle preuve de la conjecture de Baum-Connes pour les actions de complexité dynamique finie. Nous avons essayé de rendre l'article aussi indépendant du reste de la littérature que possible, afin qu'il reste accessible pour quelqu'un n'ayant suivi qu'un premier cours de K-théorie opératorielle. En particulier, nous ne supposons aucune connaissance préalable de la K-théorie contrôlée, et nous utilisons un nouveau modèle concret pour la conjecture de Baum-Connes à coefficients qui n'utilise pas la K-théorie bivariante de Kasparov.

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CONTENTS

CHAPTER 1

INTRODUCTION

Throughout this paper, the symbol '$\Gamma \circlearrowright X$' will mean that Γ is a countable discrete group, X is a compact Hausdorff space, and Γ acts on X by homeomorphisms. We will abbreviate this information by saying that '$\Gamma \circlearrowright X$ is an action'.

Our work here is based around a new property for actions, which we call *finite dynamical complexity*. This is partly inspired by the geometric notion of *finite decomposition complexity*, introduced by the first and third authors together with Tessera [**8**], and by the notion of *dynamic asymptotic dimension*, which was introduced by the current authors in earlier work [**10**].

The precise definition of finite dynamical complexity requires groupoid language to state; rather than get into details here, we just give an idea and refer the reader to Definition 3.14 (see also Definition A.4) for the precise version. Roughly, then, we say an action $\Gamma \circlearrowright X$ *decomposes* over some collection $\mathcal{C}$ of 'dynamical systems' (more precisely, étale groupoids) if it can be 'locally cut into two pieces', each of which is in $\mathcal{C}$. The action $\Gamma \circlearrowright X$ has *finite dynamical complexity* if it is contained in the smallest class $\mathcal{C}$ that is: closed under decompositions; and contains all dynamical systems that are 'essentially finite' (more precisely, have compact closure inside the ambient étale groupoid).

This definition allows the K-theory groups $K_*(C(X) \rtimes_r \Gamma)$ to be computed, at least in principle: the idea is that one can often compute the K-theory of essentially finite pieces using classical ('commutative') techniques from algebraic topology and the theory of type I C^*-algebras, then use generalized ('controlled' [**19**]) Mayer-Vietoris arguments to reassemble this into the K-theory of the whole crossed product $C(X) \rtimes_r \Gamma$. Strikingly, the C^*-algebras $C(X) \rtimes_r \Gamma$ to which these methods apply are often simple; thus one has no hope of applying classical Mayer-Vietoris techniques, as these require the presence of non-trivial ideals. This strategy works particularly well when one is trying to show vanishing of certain K-theory groups.

To illustrate this strategy for computing K-theory, the main part of this paper applies the idea above to the Baum-Connes conjecture for an action $\Gamma \circlearrowright X$ with finite dynamical complexity. This conjecture (a special case of the Baum-Connes conjecture for Γ with coefficients [4]) posits that a particular *assembly map*

$$\mu : KK_*^{\text{top}}(\Gamma, C(X)) \to K_*(C(X) \rtimes_r \Gamma) \tag{1.1}$$

is an isomorphism; here the domain is a topologically defined group associated to the action, and the codomain is the operator K-theory of the reduced crossed product C^*-algebra $C(X) \rtimes_r \Gamma$, an analytically defined object. The existence of such an isomorphism relating two quite different aspects of the action has important consequences for both: for example, it has consequences for Novikov-type conjectures associated to Γ, and implies the existence of various tools to better understand the K-theory of the crossed product.

The main part of the paper proves the following result, which is inspired in part by the third author's work [38] on the coarse Baum-Connes conjecture for spaces with finite asymptotic dimension, the first and third authors' work with Tessera on the bounded Borel conjecture for spaces with finite decomposition complexity [8], and the work of all three authors on dynamic asymptotic dimension [10].

Theorem 1.1. — *Let $\Gamma \circlearrowleft X$ be an action with finite dynamical complexity, where X is a second countable compact space. Then the Baum-Connes conjecture holds for Γ with coefficients in $C(X)$.*

Our proof of Theorem 1.1 starts by replacing the problem of proving that μ as in line (1.1) above is an isomorphism with the problem of showing that the K-theory of a certain *obstruction C^*-algebra* $A(\Gamma \circlearrowleft X)$ vanishes. For this obstruction C^*-algebra one can apply the strategy for computing K-theory outlined above, and show that it is indeed zero.

The hypotheses of Theorem 1.1 cover many interesting actions: we refer the reader to our companion paper [10], particularly the introduction, for a discussion of the case of finite dynamic asymptotic dimension. We suspect that finite dynamic dimension implies finite dynamical complexity, but did not seriously pursue that problem.

Relating the above to the literature, we should note that Theorem 1.1 is implied by earlier work: indeed, it follows from work of Tu [31] on the Baum-Connes conjecture for amenable groupoids and the fact (Theorem A.3 below) that finite dynamical complexity of a groupoid implies amenability. Some of the key tools in Tu's proof are the Dirac-dual-Dirac method of Kasparov [16], the work of Higson and Kasparov on the Baum-Connes conjecture for a-T-menable groups [11], and Le Gall's groupoid-equivariant bivariant K-theory [18]. As already hinted at above, our proof is quite different: it gives a direct way of understanding the group $K_*(C(X) \rtimes_r \Gamma)$ that uses much less machinery.

Our motivations for giving a new proof of Theorem 1.1 are fourfold. First, we want to illustrate the controlled methods for computing K-theory as already mentioned above. Second, we want to make the Baum-Connes theory more direct so that it might be adapted to computations of K-theory for much more general classes of C^*-algebras with an eye on the Künneth theorem and UCT problem as pursued in [20, 34] and [36] respectively. Third, we want to make techniques from the Baum-Connes theory more algebraic, so as to highlight and strengthen interactions with the Farrell-Jones theory in algebraic topology [**2, 3**]. Fourth, the proof is fairly self-contained: we have tried to make it accessible to a reader who has understood an introduction to C^*-algebra K-theory at the level of [**27**] or [**32**].

On this fourth point, we hope that the paper can be read without prior knowledge of Baum-Connes theory, groupoids, controlled K-theory, or even crossed product C^*-algebras. This makes the proof more elementary than most existing proofs of special cases of the Baum-Connes conjecture. In order to do this, we introduce a direct geometric / combinatorial reformulation of the Baum-Connes conjecture; we show that it agrees with the traditional one using Kasparov's KK-theory [**4**] in an appendix. Using these elementary methods also has the advantage that Theorem 1.1 remains true (correctly interpreted) if one drops the second countability assumption on X.

To conclude this introduction, we should note that this paper only just starts the study of finite dynamical complexity and its relation to other properties. We ask several open questions in A.14 through A.19 below: some of these might be difficult, but we suspect some are quite accessible.

Outline of the paper. — Section 2 builds a concrete model for the Baum-Connes assembly map for an action $\Gamma \circlearrowright X$ based on the localization algebras used by the third author to give a model for the coarse Baum-Connes assembly map [**37**]. Section 3 introduces some language from groupoid theory that will be useful in carrying out various decompositions, and which is crucial for the definition of finite dynamical complexity given at the end of that section. Section 4 gives a self-contained description of the controlled K-theory groups we will need for the proof, following work of the third author [**38**], and of Oyono-Oyono in collaboration with the third author [**19**]. Section 5 lays out the strategy for proving Theorem 1.1, which is based roughly on the proof of the coarse Baum-Connes conjecture for spaces with finite asymptotic dimension of the third author [**38**], and the work of the first and third authors with Tessera [**8**] on the stable Borel conjecture; in particular, it reduces the proof to two technical propositions. These technical propositions are established in Sections 6 and 7. There are two appendices, which require a bit more background of the reader. Appendix A relates our finite dynamical complexity to finite decomposition complexity in the sense of [**8**], and to topological amenability [**1**] as well as asking some questions; this requires some background in the general theory of étale groupoids. Appendix B identifies our model for the Baum-Connes assembly map with one of the standard models using KK-theory; as such, it requires some background in equivariant KK-theory. The appendices are included to connect what we have done here to preexisting theory, and are certainly not needed to understand the rest of the paper.

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