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BIRATIONAL GEOMETRY OF QUADRICS

by Burt Totaro

Abstract. — We construct new birational maps between quadrics over a field. The maps apply to several types of quadratic forms, including Pfister neighbors, neighbors of multiples of a Pfister form, and half-neighbors. One application is to determine which quadrics over a field are ruled (that is, birational to the projective line times some variety) in a larger range of dimensions. We describe ruledness completely for quadratic forms of odd dimension at most 17, even dimension at most 10, or dimension 14. The proof uses a new structure theorem for 14-dimensional forms, generalizing Izhboldin's theorem on 10-dimensional forms. We also show that Vishik's 16-dimensional form is ruled.

Résumé (*La géométrie birationnelle des quadriques*). — Nous construisons de nouvelles applications birationnelles entre quadriques sur un corps. Diverses formes quadratiques sont considérées: les voisines de Pfister, les voisines des multiples d'une forme de Pfister, et les demi-voisines. Un corollaire est la détermination des quadriques réglées (c'est-à-dire birationnelles au produit de la droite projective et d'une variété) en certaines dimensions. Nous décrivons complètement les quadriques réglées lorsque la forme quadratique est de dimension impaire inférieure à 17, de dimension paire inférieure à 10, ou de dimension 14. La preuve utilise un nouveau théorème de structure sur les formes de dimension 14, généralisant le théorème d'Izhboldin sur les formes de dimension 10. Nous montrons également que la forme de Vishik de dimension 16 est réglée.

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A central method in the theory of quadratic forms is the study of function fields of projective quadrics. In particular, it is important to ask when there is a rational map from one quadric over a field to another. This suggests the problem of determining when two quadrics are birational, which turns out to be much harder. The answer is known for quadratic forms of dimension at most 7 (thus for projective quadrics of dimension at most 5), by Ahmad-Ohm and Roussey among others [16, 1].

In particular, there is a conjectural characterization of which quadratic forms are ruled (meaning that the associated quadric is birational to $\mathbf{P}^1$ times some variety), Conjecture 1.1. The conjecture was known for quadratic forms of dimension at most 9 [17]. One result of this paper is to prove Conjecture 1.1 for odd-dimensional forms of dimension at most 17, and also for forms of dimension 10 or 14 (Theorem 4.1). We use in particular a new structure theorem on 14-dimensional forms (Theorem 4.2), generalizing Izhboldin's theorem on 10-dimensional forms. Vishik gave an example of a 16-dimensional form with first Witt index 2 which is not divisible by a binary form; such a form should be ruled by the conjecture, but it fell outside previously known classes of ruled forms. Nonetheless, we give a new construction which shows that Vishik's form is ruled (Theorem 7.2).

In this paper we also solve the problem of birational classification for a significant class of quadratic forms, Pfister neighbors of dimension at most 16 (Corollary 3.2). (Among these forms, only the case of "special Pfister neighbors" was known before, which fails to include all Pfister neighbors in dimensions 9, 10, and 11.) We also give several generalizations for a broader class of quadratic forms: neighbors of multiples of a Pfister form (Theorems 3.1 and 6.3), strengthening Roussey's theorem in this direction. We also give the first construction of a birational map between two non-isomorphic half-neighbors in section 5. This is genuinely different from any previously constructed birational map between quadrics.

Thanks to Alexander Vishik for allowing me to include his example of a 16-dimensional form with splitting pattern $(2, 2, 2, 2)$ which is not divisible by a binary form (Lemma 7.1).

1. Known low-dimensional results

Throughout we work over fields of characteristic not 2.

We say that two varieties over a field k are birational if they are birational over k. In particular, a quadric (meaning a smooth projective quadric hypersurface) over k is rational (birational to projective space over k) if and only if it is *isotropic*, meaning that it has a k-rational point. The proof is standard (stereographic projection). So the nontrivial problems of birational geometry for quadrics are concerned only with anisotropic quadrics, and we often tacitly assume that the quadrics we consider are anisotropic. In particular, the whole subject would be vacuous over an algebraically closed field.

A quadratic form q (always assumed nondegenerate) on an n-dimensional vector space over k determines a projective quadric $Q = \{q = 0\}$ of dimension $n-2$ in $\mathbf{P}^{n-1}$. The function field $k(q)$ means the field $k(Q)$ of rational functions on Q. We say that two quadratic forms q_1 and q_2 are *similar* if q_1 is isomorphic to aq_2 for some a in k^*. Two forms are similar if and only if the corresponding projective quadrics are isomorphic. The group of *similarity factors* $G(q)$ is the group of elements a in k^* with q isomorphic to aq. Every form can be written as a diagonal form $\langle a_1, \ldots, a_n\rangle = a_1x_1^2 + \cdots + a_nx_n^2$ for some $a_i \in k^*$. The *discriminant* $\det_\pm q$ of a quadratic form $q = \langle a_1, \ldots, a_n\rangle$ is $(-1)^{n(n-1)/2}a_1 \cdots a_n$ in $(k^*)/(k^*)^2 = H^1(k, \mathbf{F}_2)$.

We recall the definition of the splitting pattern of a quadratic form q. Let $k_0 = k$, $q_0 = q_{\rm an}$ (the anisotropic part of q, well-defined up to isomorphism), and then inductively define $k_j = k_{j-1}(q_{j-1})$ and $q_j = ((q_{j-1})_{k_j})_{\rm an}$. We stop when q_h has dimension at most 1. Then the *splitting pattern* $\mathbf{i}(q)$ is the sequence $(i_1, \ldots, i_h)$ where i_j is the Witt index $i_W((q_{j-1})_{k_j})$. The *splitting tower* of q is the sequence of fields $k_0, \ldots, k_h$. For the associated projective quadric Q, we write $i_j(Q) = i_j(q)$.

We say that two varieties over a field k, possibly of different dimensions, are *stably birational* if their products with some projective spaces over k are birational. For quadrics X and Y, a simple argument shows that stable birational equivalence is equivalent to the apparently weaker condition that X is isotropic over the function field of Y and vice versa, or equivalently that there are rational maps from X to Y and from Y to X over k [15, Theorem X.4.25]. We say that two quadratic forms over k are birational if the associated projective quadrics are birational.

By Karpenko and Merkurjev, we know exactly when an anisotropic quadric X is stably birational to a variety of lower dimension; this happens if and only if the first Witt index $i_1(X)$ is greater than 1 [10, Theorem 3.1]. (For the quadric X associated to an anisotropic quadratic form q, the first Witt index $i_1(X)$ is the maximal dimension of an isotropic linear subspace for q over the function field $k(X)$.) Every quadric X is stably birational to any subquadric Y of codimension at most $i_1(X) - 1$. This suggests the following conjecture, which would imply in particular that an anisotropic quadric is ruled (birational to $Y \times \mathbf{P}^1$ for some variety Y over k) if and only if its first Witt index is greater than 1.

Conjecture 1.1 (Ruledness conjecture). — *Let X be an anisotropic quadric over a field. Then X is birational to $Y \times \mathbf{P}^{i_1(X)-1}$ for some subquadric Y of codimension $i_1(X)-1$.*

Before this paper, Conjecture 1.1 was known for quadratic forms of dimension at most 9 [17, Lemma 3.3]. A natural companion to Conjecture 1.1 is the quadratic Zariski problem:

Conjecture 1.2 (Quadratic Zariski problem). — *If two quadrics of the same dimension over a field are stably birational, then they are birational.*

Conjecture 1.2 is known for quadratic forms of dimension at most 7, as we will discuss. A general reference on Conjecture 1.2 is [5, Theorem 6.1].

Here are the known results about birational geometry of low-dimensional quadrics in more detail; throughout we consider only anisotropic quadrics over a field. Two anisotropic conics are birational if and only they are isomorphic. Likewise, two anisotropic quadric surfaces are birational if and only if they are isomorphic, by Wadsworth [15, Theorem XII.2.2]. This hides a special phenomenon, however: a Pfister quadric surface is birational to $\mathbf{P}^1$ times any conic contained in it, by Knebusch [11, pp. 73–74]. (For example: the quadric surface $x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 = 0$ is birational to $\mathbf{P}^1$ times the conic $x_0^2 - ax_1^2 - bx_2^2 = 0$, for any a, b in k^*.) A non-Pfister quadric surface X (there are many equivalent conditions: X has first Witt index 1, or Picard number 1, or nontrivial discriminant) is not stably birational to any curve. For the definitions of Pfister forms and Pfister neighbors, see section 2.

We state Hoffmann's bound on the first Witt index; several proofs are given in the book by Elman-Karpenko-Merkurjev [3, Theorem 22.5, Corollary 26.6]. The bound is optimal, because equality holds for Pfister neighbors.

Lemma 1.3. — *An anisotropic quadratic form of dimension $2^r + s$, for $1 \leq s \leq 2^r$, has first Witt index at most s.*

All 5-dimensional anisotropic forms (corresponding to projective quadrics of dimension 3) have first Witt index 1 by Lemma 1.3, and so they are not stably birational to any lower-dimensional variety. Two non-isomorphic 5-dimensional anisotropic forms are stably birational if and only if they are both neighbors of the same Pfister 3-form, by Hoffmann [4, main theorem]. In that case, the two quadrics are in fact birational, by Ahmad-Ohm [1, Corollary 1.6, Theorem 2.5]. Lam's textbook explains Ahmad-Ohm's birational map between the quadrics $\langle 1, a, b, ab, c\rangle$ and $\langle 1, a, c, ac, b\rangle$ [15, Theorem XII.2.15]. The map was generalized by Roussey [16, Proposition 7.2.1], and generalized further in this paper (Lemma 5.1).