Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 137
Fascicule 2

2009

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 277-295

Bull. Soc. math. France
137 (2), 2009, p. 277–295

BATALIN-VILKOVISKY ALGEBRA STRUCTURES ON HOCHSCHILD COHOMOLOGY

BY LUC MENICHI

ABSTRACT. — Let M be any compact simply-connected oriented d-dimensional smooth manifold and let $\mathbb{F}$ be any field. We show that the Gerstenhaber algebra structure on the Hochschild cohomology on the singular cochains of M, $HH^*(S^*(M), S^*(M))$, extends to a Batalin-Vilkovisky algebra. Such Batalin-Vilkovisky algebra was conjectured to exist and is expected to be isomorphic to the Batalin-Vilkovisky algebra on the free loop space homology on M, $H_{*+d}(LM)$ introduced by Chas and Sullivan. We also show that the negative cyclic cohomology $HC^*_-(S^*(M))$ has a Lie bracket. Such Lie bracket is expected to coincide with the Chas-Sullivan string bracket on the equivariant homology $H^{S^1}_*(LM)$.

RÉSUMÉ (*Structures d'algèbres de Batalin-Vilkovisky sur la cohomologie de Hochschild*)

Soit M une variété lisse orientée compact simplement connexe de dimension d. Soit $\mathbb{F}$ un corps commutatif quelconque. Nous montrons que la structure d'algèbre de Gerstenhaber sur la cohomologie de Hochschild des cochaînes singulières de M, $HH^*(S^*(M), S^*(M))$, s'étend en une algèbre de Batalin-Vilkovisky. L'existence d'une telle algèbre de Batalin-Vilkovisky était conjecturée. Il est prévu qu'une telle algèbre soit isomorphe à l'algèbre de Batalin-Vilkovisky sur l'homologie des lacets libres sur M, $H_{*+d}(LM)$, introduite par Chas and Sullivan. Nous montrons aussi que la cohomologie cyclique négative $HC^*_-(S^*(M))$ possède un crochet de Lie. Ce crochet de Lie devrait coincider avec le crochet des cordes de Chas et Sullivan sur l'homologie équivariante $H^{S^1}_*(LM)$.

Texte reçu le 12 mai 2008, révisé le 28 octobre 2008, accepté le 13 novembre 2008

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2000 Mathematics Subject Classification. — 16E40, 16E45, 55P35, 57P10.

Key words and phrases. — String topology, Batalin-Vilkovisky algebra, Gerstenhaber algebra, Hochschild cohomology, free loop space.

1. Introduction

Except where specified, we work over an arbitrary field $\mathbb{F}$. Let M be a compact oriented d-dimensional smooth manifold. Denote by $LM := map(S^1, M)$ the free loop space on M. Chas and Sullivan [1] have shown that the shifted free loop homology $H_{*+d}(LM)$ has a structure of Batalin-Vilkovisky algebra (Definition 8). In particular, they showed that $H_{*+d}(LM)$ is a Gerstenhaber algebra (Definition 7). On the other hand, let A be a differential graded (unital associative) algebra. The Hochschild cohomology of A with coefficients in A, $HH^*(A, A)$, is a Gerstenhaber algebra. These two Gerstenhaber algebras are expected to be related:

Conjecture 1 (due to [1, "dictionary" p. 5] or [4]?). — *If M is simply connected then there is an isomorphism of Gerstenhaber algebras $H_{*+d}(LM) \cong HH^*(S^*(M), S^*(M))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on M.*

Félix, Thomas and Vigué-Poirrier [13, Section 7] proved that there is a linear isomorphism of lower degree d (See notation 6 for our degree conventions).

$$\mathbb{D} : HH^{-p-d}(S^*(M), S^*(M)^\vee) \overset{\cong}{\to} HH^{-p}(S^*(M), S^*(M)). \tag{2}$$

We prove

Theorem 3 (Theorem 22). — *The Connes coboundary map on $HH^*(S^*(M), S^*(M)^\vee)$ defines via the isomorphism (2) a structure of Batalin-Vilkovisky algebra extending the Gerstenhaber algebra $HH^*(S^*(M), S^*(M))$.*

Assume that M is simply-connected. Jones [19] proved that there is an isomorphism

$$J : H_{p+d}(LM) \overset{\cong}{\to} HH^{-p-d}(S^*(M), S^*(M)^\vee)$$

such that the Δ operator of the Batalin-Vilkovisky algebra $H_{*+d}(LM)$ and Connes coboundary map $B^\vee$ on $HH^{*-d}(S^*(M), S^*(M)^\vee)$ satisfies $J \circ \Delta = B^\vee \circ J$. Of course, we conjecture:

Conjecture 4. — *The isomorphism*

$$\mathbb{D} \circ J : H_{p+d}(LM) \overset{\cong}{\to} HH^{-p}(S^*(M), S^*(M))$$

is a morphism of graded algebras.

Notice that Conjecture 4 implies that the composite $\mathbb{D} \circ J$ is an isomorphism of Batalin-Vilkovisky algebras between the Chas-Sullivan Batalin-Vilkovisky algebra and the Batalin-Vilkovisky algebra defined by Theorem 22. Therefore Conjecture 4 implies Conjecture 1.

Cohen and Jones [4, Theorem 3] first mentioned an isomorphism of algebras

$$H_{p+d}(LM) \stackrel{\cong}{\to} HH^{-p}(S^*(M), S^*(M)).$$

Over the reals or over the rationals, two proofs of such an isomorphism of graded algebras have been given by Merkulov [25] and Félix, Thomas, Vigué-Poirrier [14].

Theorem 22 comes from a general result (Propositions 11 and 12) which shows that the Hochschild cohomology $HH^*(A, A)$ of a differential graded algebra A which is a "symmetric algebra in the derived category", is a Batalin-Vilkovisky algebra. As second application of this general result, we recover the following theorem due to Thomas Tradler.

Theorem 5 ([26, Example 2.15 and Theorem 3.1] (Corollary 19))
Let A be a symmetric algebra. Then $HH^*(A, A)$ is a Batalin-Vilkovisky algebra.

This theorem has been reproved and extended by many people [5, 8, 18, 20, 21, 22, 23, 28] (in chronological order). The last proof, the proof of Eu et Schedler [8] looks similar to ours.

Thomas Tradler gave a somehow complicated proof of the previous theorem (Corollary 19). Indeed, his goal was to prove our main theorem (Theorem 22). In [29] or in [27], Tradler and Zeinalian proved Theorem 22 but only over a field of characteristic 0 [29, "rational simplicial chain" in the abstract] or [27, Beginning of 3.1]. Costello's result [5, Section 2.1] is also over a field of characteristic 0.

Over $\mathbb{Q}$, we explain in Corollary 20 how to put a Batalin-Vilkovisky algebra structure on $HH^*(S^*(M; \mathbb{Q}), S^*(M; \mathbb{Q}))$ from a slight generalisation of Corollary 19 (Theorem 18). In fact both Félix, Thomas [12] and Chen [3, Theorem 5.4] proved that the Chas-Sullivan Batalin-Vilkovisky algebra $H_{*+d}(LM; \mathbb{Q})$ is isomorphic to the Batalin-Vilkovisky algebra given by Corollary 20.

Remark that, over $\mathbb{Q}$, when the manifold M is formal, a consequence of Félix and Thomas work [12], is that $H_{*+d}(LM)$ is always isomorphic to the Batalin-Vilkovisky algebra $HH^*(H^*(M); H^*(M))$ given by Corollary 19 applied to the symmetric algebra $H^*(M)$. Over $\mathbb{F}_2$, in [24], we showed that this is not the case. The present paper seems to explain why:

The Batalin-Vilkovisky algebra on $HH^*(S^*(M), S^*(M))$ given by Theorem 22 depends of course on the algebra $S^*(M)$ but also on a fundamental class $[m] \in HH^{-d}(S^*(M), S^*(M)^\vee)$ which seems hard to compute. This fundamental class $[m]$ involves chain homotopies for the commutativity of the algebra $S^*(M)$.

The Batalin-Vilkovisky algebra on $HH^*(S^*(M;\mathbb{Q}), S^*(M;\mathbb{Q}))$ given by Corollary 20, depends of

- a commutative algebra, Sullivan's cochain algebra of polynomial differential forms $A_{PL}(M)$ [10],
- and of the fundamental class $[M] \in H_d(A_{PL}(M)^\vee)$.

Acknowledgment. — We wish to thank Jean-Claude Thomas for a discussion concerning Ginzburg's preprint [17]. We would like also to thank Yves Félix for explaining us the wonderful isomorphism (2).

2. Hochschild homology and cohomology

We use the graded differential algebra of [10, Chapter 3]. In particular, an element of lower degree $i \in \mathbb{Z}$ is by the *classical convention* [10, p. 41-2] of upper degree $-i$. Differentials are of lower degree -1. All the algebras considered in this paper, are unital and associative. Let A be a differential graded algebra. Denote by sA the suspension of A, $(sA)_i = A_{i-1}$. Let d_0 be the differential on the tensor product of complexes $A \otimes T(sA) \otimes A$. We denote the tensor product of the elements $a \in A$, $sa_1 \in sA$, $\dots$, $sa_k \in sA$ and $b \in A$ by $a[a_1|\cdots|a_k]b$. Let d_1 be the differential on the graded vector space $A \otimes T(sA) \otimes A$ defined by:

$$\begin{aligned} d_1 a[a_1|\cdots|a_k]b = (-1)^{|a|} & a a_1[a_2|\cdots|a_k]b \\ & + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[a_1|\cdots|a_i a_{i+1}|\cdots|a_k]b \\ & - (-1)^{\varepsilon_{k-1}} a[a_1|\cdots|a_{k-1}]a_k b; \end{aligned}$$

Here $\varepsilon_i = |a| + |a_1| + \cdots + |a_i| + i$. The *bar resolution of* A, denoted $B(A;A;A)$, is the differential graded (A,A)-bimodule $(A \otimes T(sA) \otimes A, d_0 + d_1)$.

Denote by A^{op} the opposite algebra of A. Recall that any (A,A)-bimodule can be considered as a left (or right) $A \otimes A^{\mathrm{op}}$-module. The *Hochschild chain complex* is the complex $A \otimes_{A \otimes A^{\mathrm{op}}} B(A;A;A)$ denoted $\mathcal{C}_*(A,A)$. Explicitly $\mathcal{C}_*(A,A)$ is the complex $(A \otimes T(sA), d_0 + d_1)$ with d_0 obtained by tensorization