Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

LARGE DEVIATIONS AND PATH PROPERTIES OF THE TRUE SELF-REPELLING MOTION

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Tome 146
Fascicule 1

2018

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 215-240

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 146, mars 2018

Diffusion

Maison de la SMF
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France
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AMS
P.O. Box 6248
Providence RI 02940
USA
`http://www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

Bull. Soc. Math. France
146 (1), 2018, p. 215-240

LARGE DEVIATIONS AND PATH PROPERTIES OF THE TRUE SELF-REPELLING MOTION

by Laure Dumaz

Abstract. — We derive some large deviation bounds for events related to the "true self-repelling motion," a one-dimensional self-interacting process introduced by Tóth and Werner, that has very different path properties than usual diffusion processes. We then use these estimates to study certain of these path properties such as its law of iterated logarithms for both small and large times.

Résumé (*Grandes déviations et propriétés trajectorielles du « vrai » processus auto-répulsif*). — Nous montrons dans cet article certaines bornes de grandes déviations pour des événements liés au « vrai » processus auto-répulsif, un processus unidimensionnel introduit par Toth et Werner, qui a des propriétés trajectorielles très différentes de celles des diffusions usuelles. Nous utilisons ensuite ces estimées pour étudier certaines de ces propriétés trajectorielles concernant la loi du logarithme itéré pour les petits temps ainsi que les grands temps.

1. Introduction

In the present paper, we study some features of a self-interacting one-dimensional process called the true self-repelling motion, defined by Tóth and

Texte reçu le 3 août 2012, modifié le 19 avril 2013, accepté le 17 mai 2013.

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Mathematical subject classification (2010). — 60G18, 60K35, 60G17, 60F10.

Key words and phrases. — True self-repelling motion, Brownian web, self-interacting processes, local time, large deviations, law of the iterated logarithm.

 0037-9484/2018/215/$ 5.00
 doi:10.24033/bsmf.2756

Werner in [9]. Let us first very briefly recall the intuitive definition of this process and describe the motivations that lead to our study.

The true self-repelling motion is a continuous real-valued process $(X_t, t \geq 0)$ that is locally self-interacting with its past occupation-time. More precisely, for each positive time t, define its occupation-time measure μ_t that assigns to each interval $I \subset \mathbb{R}$, the time spent in it by X before time t:

$$\mu_t(I) = \int_0^t 1_{X_s \in I} \, ds.$$

It turns out that for this particular process X, almost surely for each t, the measure μ_t has a continuous density $L_t(x)$. By analogy with semi-martingales, where such occupation-time densities also exist, the curve $x \mapsto L_t(x)$ is called the "local-time" profile of X at time t. Heuristically, the dynamics of X_t is such that the TSRM is locally pushed in the direction of the negative "gradient" of the local time at its current position. Loosely formulated, one can write $dX_t = -\nabla_x L_t(X_t)dt$ (even if $(X_t, t \geq 0)$ is a random process). For more details and comments on this description, we refer to [9]. It turns out that this process is of a very different type than diffusions. For example (see again [9]), its quadratic variation almost surely vanishes whereas its variation of power $3/2$ is positive and finite. Similarly, it does not have the Brownian scaling property, it has instead a $2/3$ scaling behavior i.e., for any positive λ, $(X_{\lambda t}, t \geq 0)$ has the same law as $(\lambda^{2/3} X_t, t \geq 0)$.

This same exponent $2/3$ appears in various other models that can be interpreted as continuous height-fluctuations of $1+1$-dimensional models in the Khardar-Parisi-Zhang universality class (such as the Tracy-Widom distribution for eigenvalues of large random matrices, the movement of the second-class particle in a TASEP etc.). TSRM seems however at present to be one of the few such "non-diffusive" continuous processes that probabilists can define (see also [2] for related questions). All this gives us some motivation to study in more detail its behavior, in order to see what features it shares with the other previously-mentioned models, and also for its own independent interest.

Let us now describe briefly the results of the present paper: Both for the process $(X_t, t \geq 0)$ itself as for the height process $(H_t, t \geq 0)$, we give upper and lower bounds for the probability that their value at a given time is very large. Combined with $0-1$-law arguments, this enables us to derive almost sure fluctuation results (of the type of the law of the iterated logarithm) for these two processes. For instance, we shall see that $\limsup_{t\to\infty} X_t/(t^{2/3}(\log\log t)^{1/3})$ is almost surely equal to a finite positive constant, and a similar result when $t \to 0$.

The construction of the process X_t is based on a family of coalescing one-dimensional Brownian motions starting from all points in the plane. Such families had been constructed by Arratia in [1], and further studied in [9, 8, 3, 6] and are called "Brownian web" in the latter papers. As a consequence, the estimates on the TSRM will follow from results concerning this Brownian web. In Section 2, we will recall some aspects of the construction of TSRM and some features of the Brownian web. In Section 3, we will focus on the large deviation estimates concerning X_1, we then derive the LIL for X in Section 4, and we finally focus on the fluctuations of the height-process in the final Section 5.

Acknowledgement. — I am grateful to my supervisors Bálint Tóth and Wendelin Werner for their guidance throughout this work. Special thanks go to Wendelin Werner for his careful reading of successive versions of this paper, and to the referees for their insightful comments.

2. Preliminaries and notations

In this section, we put down some notation, and collect some elementary estimates that will be useful later on.

2.1. Versions of the Brownian web. — The true self-repelling motion (TSRM) is a deterministic function of a certain family of coalescing one-dimensional Brownian motions. There are two natural variants of TSRM, that respectively correspond to such Brownian families in the entire plane (this is the "stationary" TSRM, this version has stationary increments) or in the upper half-plane (this is the TSRM with "zero-initial conditions"). Other initial conditions are also possible, see Section 4 of [8] for examples.

Let us briefly first recall the construction in the *stationary* case which will be the main focus of this paper. To start with, choose any deterministic countable dense family Q of points $(\tilde{x}, \tilde{h})$ in the plane, say $Q = \mathbb{Q}^2$. It is then possible to define the joint law of a family $(\Lambda_{\tilde{x},\tilde{h}}(\cdot), (\tilde{x}, \tilde{h}) \in Q)$ in such a way that, for each $(\tilde{x}, \tilde{h}) \in Q$, $\Lambda_{\tilde{x},\tilde{h}}$ is a function from $[\tilde{x}, \infty)$ into $\mathbb{R}$, that is distributed like a Brownian motion started from height $\tilde{h}$ at time $\tilde{x}$. Furthermore (see e.g., [9] for details), different curves are "independent until their first meeting time" and they coalesce after this meeting time (and follow the same Brownian evolution). Recall that Q is dense in the plane, so that the picture of all these lines is dense in the plane. The coalescent structure nevertheless defines a tree-like structure rooted "at $x = +\infty$". This family of curves Λ is often referred to as the "forward lines".

If we are given a countable dense family $\tilde{Q}$ in the plane, then one can almost surely define the family of "backward" lines $(\Lambda_{\tilde{x},\tilde{h}}(\cdot), (\tilde{x}, \tilde{h}) \in \tilde{Q})$ such that each $\Lambda_{\tilde{x},\tilde{h}}$ is now a function defined on $(-\infty, \tilde{x}]$ in such a way that the backward lines can be viewed as the "dual tree" of the previous dense tree (it is therefore a deterministic function of all forward lines). It is proved in [9] that this family of backward lines has the same law as the reversed image (changing x into $-x$) of the law of the forward lines (choosing $\tilde{Q}$ to be the symmetric image of Q).