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MICROLOCALIZATION OF SUBANALYTIC SHEAVES

Luca Prelli

***Abstract.* —** We define the specialization and microlocalization functors for subanalytic sheaves. Applying these tools to the sheaves of tempered and Whitney holomorphic functions, we generalize some classical constructions. We also prove that the microlocalizations of tempered and Whitney holomorphic functions have a natural structure of module over the ring of microdifferential operators, and are locally invariant under contact transformations.

***Résumé* (Microlocalisation des faisceaux sous-analytiques). —** On définit la spécialisation et la microlocalisation pour les faisceaux sous-analytiques. En appliquant ces outils aux faisceaux des fonctions holomorphes tempérées et de Whitney, on généralise des constructions classiques. On démontre aussi que les microlocalisations des fonctions holomorphes tempérées et de Whitney ont une structure naturelle de module sur l'anneau des opérateurs microdifférentiels, et sont localement invariants par transformations de contact.

CONTENTS

INTRODUCTION

After the fundamental works of Sato on hyperfunctions and microfunctions and the development of algebraic analysis, the methods of cohomological theory of sheaves became very useful for studying systems of PDE on real or complex analytic manifolds. Motivated by the study of solutions with growth conditions of a system of PDE (Riemann-Hilbert correspondence, Laplace transform, etc.), Kashiwara and Schapira in [**16**] introduced the notion of ind-sheaf, and defined the formalism of six Grothendieck operations in this framework. They defined the subanalytic site (a site whose open sets are subanalytic and the coverings are locally finite) and obtained the ind-sheaves of tempered and Whitney holomorphic functions (which are objects of the derived category of sheaves on this site) by including subanalytic sheaves into the category of ind-sheaves. Then, in [**28**], a direct, self-contained and elementary construction of the six Grothendieck operations for subanalytic sheaves was established. Important examples of applications of subanalytic sheaves to $\mathcal{D}$-modules can be found in [**24**] and [**25**].

The microlocalization functor for sheaves on a real analytic manifold was originally introduced by Sato to perform a microlocal analysis of the singularities of hyperfunction solutions of systems of linear PDE on complex manifolds. It was generalized to the framework of ind-sheaves in [**19**]. It is natural to ask if it is possible to develop microlocalization on the subanalytic site avoiding the heavy theory of ind-sheaves. The aim of this work is to extend some classical constructions for sheaves, as the functors of specialization and microlocalization, to the framework of subanalytic sheaves.

We introduce first the category of conic subanalytic sheaves on an analytic manifold endowed with an action of $\mathbb{R}^+$. In order to do that we have to choose a suitable definition: indeed there are several definitions, which are equivalent in the classical case but not in the framework of subanalytic sheaves. We choose the one which satisfies some desirable properties, as the equivalence with sheaves on the conic topology associated to the action. Thanks to this equivalence we can also represent conic sheaves as limits of conic $\mathbb{R}$-constructible sheaves. Then we extend the Fourier-Sato transform to the category of conic subanalytic sheaves on a vector bundle. This construction was also motivated by the sheaf theoretical interpretation given in [**31**] of the Laplace isomorphisms of Kashiwara and Schapira. At this point we can start studying subanalytic sheaves from a microlocal point of view by introducing the functors of specialization and microlocalization along a submanifold of a real analytic manifold. As an interesting application, the specialization is the key tool used in order to give a functorial construction of asymptotically developable functions (see also the recent developments in [**11**]). We give an estimate of the support of microlocalization using the subanalytic analogue of the notion of ind-microsupport of [**17**] and its functorial properties developed in [**23**]. We also show that the functor of microlocalization is related with the functor of ind-microlocalization defined in [**19**]. Then, applying specialization (resp. microlocalization) to the subanalytic sheaves of tempered and Whitney holomorphic functions, we generalize tempered and formal specialization (resp. microlocalization). In this way we get a unifying description of Andronikof's [**1**] and Colin's [**6**] "ad hoc" constructions.

As an application, we prove that the microlocalizations of $\mathcal{O}^t$ and $\mathcal{O}^{\mathrm{w}}$ have (in cohomology) a natural structure of $\mathcal{E}$-module and that locally they are invariant under contact transformations. Only in the case of $\mathcal{O}^t$ these results were proven in [**1**]. Furthermore, using DG-methods and ind-microlocalization, in [**10**] the author proved that the microlocalization of tempered holomorphic functions is an object of the derived category of $\mathcal{E}$-modules. The $\mathcal{E}$-module structure, combined with the estimate for the support of microlocalization, was essential for the proof of a Cauchy-Kowalevskaya-Kashiwara theorem with growth conditions given in [**29**].

In more details the contents of this work are as follows.

In Chapter 1 we recall the results on subanalytic sheaves of [**16**] and [**28**].

In Chapter 2 we construct the category of conic sheaves on a subanalytic site endowed with an action of $\mathbb{R}^+$.

In Chapter 3 we consider a vector bundle E over a real analytic manifold and its dual E^* endowed with the natural action of $\mathbb{R}^+$. We define the Fourier-Sato transform which gives an equivalence between conic subanalytic sheaves on E and conic subanalytic sheaves on E^*.

Then we define the functor ν_M^{sa} of specialization along a submanifold M of a real analytic manifold X (Chapter 4) and its Fourier-Sato transform, the functor μ_M^{sa} of microlocalization (Chapter 5). We introduce the functor μhom^{sa} for subanalytic sheaves and we give an estimate of its support using the notion of microsupport of [**17**]. Then we study its relation with the functor of ind-microlocalization of [**19**].

We apply these results in Chapter 6. We study the connection between specialization and microlocalization for subanalytic sheaves and the classical ones. Specialization of subanalytic sheaves generalizes tempered and formal specialization of [**1**]