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Higher Castelnuovo Theory

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0. Introduction.

In this paper and others to follow, we intend to set out a series of conjectures concerning the Hilbert functions of points (or more generally, zero-dimensional subschemes) in projective space; or, more generally still, the Hilbert functions of graded Artinian rings. We were first led to make some of these conjectures in Eisenbud-Harris [1982] in the course of our work on Castelnuovo theory. A special case of these was proved independently by us in that paper and by Miles Reid – though as Ciliberto later noted [1987] we were both anticipated by G. Fano [1894]. Recently, we saw how our conjectures might be generalized; and in this form they relate to a number of other areas: for example, another special case is equivalent to a conjectured generalization of the classical Cayley-Bacharach theorem (as we will also discuss here); another to the Kruskal-Katona and Clements-Lindström theorems of combinatorics (see, for example, Kleitman-Green [1978]); and still others, which we intend to describe in a later paper, to questions about the existence of exceptional linear series on complete intersection curves.

Good references for unexplained terminology are Arbarello-Cornalba-Griffiths-Harris [1985] or Eisenbud-Harris [1982].

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1. Castelnuovo theory.

Recall that a set of points in projective space is in *uniform position* if the Hilbert function (= postulation) of a subset depends only on the cardinality of the subset. Castelnuovo theory is concerned with the possible Hilbert functions of points in uniform position. Its origins are classical: Castelnuovo first used estimates on the Hilbert functions of points to derive his upper bound on the genus of an irreducible nondegenerate curve C in projective space $\mathbf{P}^r$ in terms of the degree d of C. Castelnuovo's argument has been reproduced too many times to repeat in detail here (see, for example, Eisenbud-Harris [1982] or Arbarello-Cornalba-Griffiths-Harris [1985]), but briefly what he shows first, by completely elementary means, is that if $\Gamma \subset \mathbf{P}^{n-1}$ is a general hyperplane section of C then

$$g(C) \leq \sum_{\ell=1}^{\infty} h^1(\mathbf{P}^{n-1}, \mathcal{I}_\Gamma(\ell))$$

or, in other words, the genus of C is bounded by the sum over all ℓ of the failure of Γ to impose independent conditons on hypersurfaces of degree ℓ. Curves of maximal genus for their degree therefore are likely to be those whose hyperplane sections Γ have the smallest possible Hilbert function h_Γ. Next, Castelnuovo shows that among all configurations Γ of $d \geq 2n+1$ points in uniform position in $\mathbf{P}^{n-1}$, the ones with minimal Hilbert function are exactly those lying on rational normal curves; he calculates his bound $\pi(d,n)$ on the genus of a curve accordingly. Finally, since if Γ is a subset of a rational normal curve any quadric containing Γ will contain the rational normal curve, he shows that if C is a curve achieving his bound the quadrics containing C must cut out in $\mathbf{P}^n$ a surface whose hyperplane section is a rational normal curve (in particular, a surface of degree $n-1$, the minimum possible degree for a nondegenerate surface in $\mathbf{P}^n$).

In Eisenbud-Harris [1982], we undertook to extend the results of Castelnuovo – in particular, his characterization of curves of maximal genus for their degree as lying on rational normal scrolls – to curves of high, but not maximal genus. This involved asking, for example, "What is the second smallest possible Hilbert function of a collection of points?" and in general, "What configurations of points have small Hilbert function?" What emerged was the following philosophy: The way to achieve a configuration $\Gamma \subset \mathbf{P}^r$ in uniform position having small Hilbert function is to put Γ on a positive-dimensional variety with small Hilbert function – in effect, on a curve of smallest possible degree, and of largest possible genus given that degree – which is the intersection of the hypersurfaces of low degree containing Γ.

To be specific, let $\Gamma \subset \mathbf{P}^r$ be a nondegenerate collection of d points in uniform position; let h_Γ be its Hilbert function, so that for example $h_\Gamma(2)$ is the number of conditions imposed by Γ on quadrics. Castelnuovo says that if $d \geq 2r+3$, then Γ must impose at least $2r+1$ conditions on quadrics; and if $h_\Gamma(2) = 2r+1$ exactly, then Γ must lie on a rational normal curve. Extending this, it turned out that if $d \geq 2r+5$ and if $h_\Gamma \geq 2r+2$ then necessarily Γ had to lie on an elliptic normal curve (Fano [1894],Eisenbud-Harris [1982], Reid [unpublished]). We deduced in particular that if a curve $C \subset \mathbf{P}^n$ had genus exceeding a bound $\pi_1(d,n)$ (substantially lower than $\pi(d,n)$), then the quadrics containing C have to cut out a surface of degree n in $\mathbf{P}^n$, which allowed us to classify such curves. Both we and Miles Reid went on to conjecture that this pattern would persist, at least for a while: for $\alpha < r$, we conjectured, under the hypothesis $d \geq 2r + 2\alpha + 1$ we could conclude that either $h_\Gamma \geq 2r + \alpha + 1$ or Γ lay on a curve of degree $r + \alpha - 1$ or less in $\mathbf{P}^r$.

In all of these cases, the latter conclusion – that Γ lay on a curve of small degree – would follow immediately if one knew that the intersection of the quadrics containing Γ was in fact positive dimensional. This observation last year suggested to us a seemingly trivial restatement. If we hypothesize that Γ is cut out by quadrics, we can ask: given $h_\Gamma(2)$, what is the largest possible d? In other words, *What is the largest number $d(h)$ of points of intersection of a linear system of quadrics of codimension h in the space of all quadrics in $\mathbf{P}^r$, given that the intersection of those quadrics is zero-dimensional?* In these terms, we may summarize the state of our knowledge as of 1981 (and its origins) as follows:

$$
\begin{array}{ll}
d(r+1) = r+1 & \text{(elementary)} \\
d(r+2) = r+2 & \text{(elementary)} \\
\vdots & \\
d(2r-1) = 2r-1 & \text{(elementary)} \\
d(2r) = 2r & \text{(elementary)} \\
d(2r+1) = 2r+2 & \text{(Castelnuovo)} \\
d(2r+2) = 2r+4 & \text{(Fano, Eisenbud-Harris, Reid)}
\end{array}
$$

The conjectures mentioned above extend this pattern to:

$$d(2r+3) = 2r+6$$

$$\vdots$$

$$d(3r-3) = 4r-6$$

$$d(3r-2) = 4r-4.$$

Note that this conjectured bound on the number of points is sharp, if it holds: for $h \leq 2r$, of course, any configuration of h points in linear general position will be cut out by quadrics and will impose independent conditions on quadrics; and for $2r+2 \leq h = 2r+\alpha \leq 3r-2$ we can take Γ the interseciton of a linearly normal curve of degree $r+\alpha$ – that is, a curve of degree $r+\alpha$ and (maximal) genus α – with another quadric. Note, moreover, that in the last case – $d(3r-2) = 4r-4$ – there is also another example we can use to show that the bounds is sharp: we can take Γ the intersection of a rational normal scroll $X \subset \mathbf{P}^r$ with two more quadrics.

This last example suggests that at this point the pattern of $d(h)$ increasing by 2 each time stops. Indeed, corresponding to the two examples above in case $h = 3r-2$ there are two examples to suggest that the next value of d should be

$$d(3r-1) = 4r.$$

On the one hand, the maximal genus of a curve of degree $r+\alpha$ in $\mathbf{P}^r$ increases by 2 from $\alpha = r-1$ to $\alpha = r$, with the result that a curve of degree $2r-1$ and genus $r-1$ in $\mathbf{P}^r$ will lie on the same number of quadrics as a curve of degree $2r$ and genus $r+1$ (that is, a canonical curve). Thus we can take Γ the intersection of a canonical curve in $\mathbf{P}^r$ with a quadric to arrive at a configuration of $4r$ points imposing only $3r-1$ conditions on quadrics. On the other hand, in the latter example, if we replace the rational normal surface scroll S, which has degree $r-1$, with a linearly normal surface of one larger degree r (for example, a del Pezzo surface or a cone over an elliptic normal curve), the intersection of our surface with two quadrics will again have degree $4r$ and impose $3r-1$ conditions on quadrics.

Similar examples indicate that for the next $r-3$ steps $d(h)$ will increase by 4 each time we increase h: by way of an example, we can take Γ the intersection of a surface of degree $r-1+\beta$ with two further quadrics. When we get to the case $h = 4r-5$, however, we get a new example: the intersection