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FOUNDATIONS OF TWISTED ENDOSCOPY

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Abstract. — This book develops the foundations of a general theory of twisted endoscopy: definition of endoscopic groups, study of the correspondance between twisted conjugacy classes and conjugacy classes in endoscopic groups, definition of transfer factors, and finally the stabilization of the elliptic part of the twisted trace formula. The book also develops a theory of duality and Tamagawa numbers for the hypercohomology of complexes $T \to U$ of tori.

***Résumé* (Fondements de l'endoscopie tordue).** — Ce livre développe les bases de la théorie générale de l'endoscopie tordue: définitions des groupes endoscopiques, étude de la correspondance entre classes de conjugaison tordue et classes de conjugaison sur un groupe endoscopique, définition du facteur de transfert, enfin stabilisation de la partie elliptique de la formule des traces tordue. Le livre développe également une théorie de dualité et de nombres de Tamagawa pour l'hypercohomologie de complexes $T \to U$ de tores.

Contents

INTRODUCTION

In this paper we begin a study of the foundations for a theory of twisted endoscopy for reductive groups. While we build on standard endoscopy as developed, for example, in **[K1]**, **[K3]**, **[L2]**, **[LS1]** and **[S1]**, there are new features which will be described below and which in turn shed further light on the earlier theory.

In our setting F is a local or global field of characteristic zero, G is a connected reductive group defined over F, θ is an F-automorphism of G and ω is a quasicharacter on $G(F)$ if F is local or on $G(\mathbb{A})$ trivial on $G(F)$ if F is global. Endoscopy for (G, θ, ω) concerns the representations π of $G(F)$ or $G(\mathbb{A})$, as appropriate, for which $\pi \circ \theta$ is equivalent to $\omega \otimes \pi$. More generally, and for the most part conjecturally, we may consider L-packets or Arthur packets Π for which $\Pi \circ \theta = \omega \otimes \Pi$. Associated with such representations is a (θ, ω)-twisted invariant harmonic analysis: an Arthur trace formula, (θ, ω)-twisted characters, (θ, ω)-twisted orbital integrals and so on. Twisted endoscopy has played a role in a variety of problems. For example, the early paper **[LL]** of Labesse and Langlands on standard endoscopy for $\mathbf{SL}(2)$ is at the same time a study of a twisted endoscopy problem for $\mathbf{GL}(2)$: $\pi = \omega \otimes \pi$, and in the study of automorphic representations of unitary groups in three variables **[R]** we find the twisted endoscopy associated with base change.

We will begin by introducing endoscopic groups, or better endoscopic data, for $(G, \theta, \mathbf{a})$, where $\mathbf{a}$ is a Langlands parameter for ω. Our definitions were announced several years ago and indeed were used to recast the definitions for standard endoscopy in **[LS1]**. What remains perhaps as a surprise is the effort required in the general case to accommodate the possible lack of a suitable embedding of the L-group of an endoscopic group in the L-group of G. The basic theme of endoscopy is transfer from H to G, or more properly, transfer from a z-extension H_1 of H to G. At the level of F- or $\mathbb{A}$-points on the groups, examples such as base change or symmetric square for $\mathbf{GL}(2)$ have relied on concretely defined norm mappings. For the general case we take another more abstract approach, one which is well adapted to arguments involving the relevant systems of roots and restricted roots.

Now suppose that F is local. With the notion of norm mappings from sufficiently regular classes of elements in $G(F)$ to classes in $H_1(F)$ we can turn to the matching of (θ, ω)-twisted orbital integrals on $G(F)$ with stable orbital integrals on $H_1(F)$. The first goal of this paper will be to construct *transfer factors.* In analogy with standard endoscopy **[LS1]** these are the weighting factors for the (θ, ω)-twisted integrals needed to achieve the matching with the integrals on $H_1(F)$. Again as in the standard case they are quite elaborate for they must carry a great deal of information about the values of characters on the groups $G(F)$ and $H_1(F)$. There are new features. We need a slight generalization of the comparative study of the embeddings of L-groups of maximal tori in the L-group of a reductive group from **[LS2]** in order to construct one of our terms. We replace the Galois cohomology of standard endoscopy with Galois hypercohomology (for some complexes of tori of length 2) and introduce a pairing on hypercohomology that encompasses both the Langlands pairing for tori and Tate-Nakayama duality. We then gather all three cohomologically defined terms Δ_{I}, Δ_1, Δ_2 from the standard case into one term involving this pairing, although for the purposes of proof of canonicity and so on we have found it convenient to write this one term as a product of two, Δ_{I} and Δ_{III}.

The first five chapters of the paper, which treat transfer factors, are organized as follows. The first chapter reviews results of Steinberg on semisimple automorphisms of semisimple groups. These results are used repeatedly in the rest of the paper, often without comment. At the end of this chapter one finds the definition of a-data and χ-data for twisted endoscopy.

The second chapter begins by giving the definition of endoscopic data $(H, \mathcal{H}, s, \xi)$ for $(G, \theta, \mathbf{a})$. The group $\mathcal{H}$ is an extension of W_F by $\widehat{H}$ and ξ is an L-homomorphism from $\mathcal{H}$ to ${}^L G$. It is not always the case that the identity map from $\widehat{H}$ to itself can be extended to an L-isomorphism from $\mathcal{H}$ to ${}^L H$, which forces us to use z-pairs (H_1, ξ_{H_1}), consisting of a z-extension H_1 of H and an L-embedding ξ_{H_1} of $\mathcal{H}$ in ${}^L H_1$ extending the natural inclusion of $\widehat{H}$ in $\widehat{H}_1$. The existence of such L-embeddings is proved in Lemma 2.2.A.

The third chapter introduces the abstract norm mapping which relates conjugacy classes in $H(\overline{F})$ and twisted conjugacy classes in $G(\overline{F})$. For this one first constructs a bijection from the set of twisted conjugacy classes in $G(\overline{F})$ to the analogous set for a quasi-split inner form of (G, θ). Unfortunately, unless the center of G is trivial, this map is not canonical, and there may in fact be no choice for it which is defined over F. For most of this paper we treat only the special case in which this difficulty does not occur (in other words we assume that the 1-cochain z_σ in (3.1) is trivial); then in (5.4) we explain the modifications needed in the general case.

The fourth chapter gives the definition of the relative transfer factor $\Delta(\gamma_1, \delta; \overline{\gamma}_1, \overline{\delta})$, which should be thought of as the ratio

$$\Delta(\gamma_1, \delta)/\Delta(\overline{\gamma}_1, \overline{\delta})$$

of absolute transfer factors, these being canonical only up to a non-zero scalar (independent of γ_1, δ of course). In the case of standard endoscopy our relative transfer factor coincides with the one in **[LS1]**, except that **[LS1]** takes z-extensions of G while our more general situation forces us to take z-extensions of H instead. The relative transfer factor is the product of four terms, the third of which, Δ_{III}, is the most complicated.

The fifth chapter uses the relative transfer factors to define absolute transfer factors $\Delta(\gamma_1, \delta)$ and lists their most important properties (see Lemmas 5.1.B, 5.1.C and Theorem 5.1.D). In case G is quasi-split and θ preserves an F-splitting $\mathbf{spl}_G$ there is a particular normalization $\Delta_0(\gamma_1, \delta)$ of the absolute transfer factor, depending only on $\mathbf{spl}_G$, just as in the standard case **[LS1]**. Let B_0 be the Borel subgroup appearing in the splitting $\mathbf{spl}_G$ and let λ be a θ-stable generic character on the F-points of the unipotent radical of B_0. Then one hopes **[Sh]** that the representations having Whittaker models for λ will serve as base points in tempered L-packets, and in (5.3) this leads us to multiply the absolute factor $\Delta_0(\gamma_1, \delta)$ by a suitable local ε-factor so as to obtain another absolute transfer factor $\Delta_\lambda(\gamma_1, \delta)$, depending only on the generic character λ. In (5.5) we give the definition of matching functions.

Before trying to understand the complicated factor Δ_{III} in the relative situation of (4.4), the reader may find it useful to study the absolute analogue of Δ_{III} given in (5.3). We now sketch this material under a number of simplying hypotheses, in the hope that the main idea will come through as clearly as possible. So let us assume for the moment that G is quasi-split, semisimple and simply connected. In particular $\mathbf{a}$ and ω are then trivial. Assume further that θ preserves some F-splitting of G. Let T be a θ-stable maximal F-torus of G that is contained in some θ-stable Borel subgroup B of G. Note that we do not assume that B is defined over F. Let T_θ denote the torus $T/(1-\theta)(T)$ (the coinvariants of θ on T). We think of the canonical surjection $N : T \to T_\theta$ as an abstract norm map. Let $\delta \in G(F)$ and $\gamma \in T_\theta(F)$, and assume that γ is sufficiently regular. We say that γ is a norm of δ if there exist elements $t \in T(\overline{F})$ and $g \in G(\overline{F})$ such that $N(t) = \gamma$ and $g\delta\theta(g)^{-1} = t$. Applying $\sigma \in \Gamma := \mathrm{Gal}(\overline{F}/F)$ to the equality $g\delta\theta(g)^{-1} = t$ and using that $N(t) = \gamma$ as well as the fact that γ is sufficiently regular (so that the twisted centralizer of t is the group of fixed points of θ on T) we see that the 1-cocycle $t_\sigma := g\sigma(g)^{-1}$ takes values in T and satisfies the equality

$$t \cdot \sigma(t)^{-1} = t_\sigma \theta(t_\sigma)^{-1}.$$

This equality simply says that the pair (t_σ^{-1}, t) is a 1-hypercocycle of Γ in the complex $T \xrightarrow{1-\theta} T$. The class $\mathrm{inv}(\gamma, \delta)$ of this 1-hypercocycle lies in the hypercohomology group $H^1(F, T \xrightarrow{1-\theta} T)$.

Now suppose that we are given a twisted endoscopic group H for (G, θ) and an admissible isomorphism over F from T_θ to a maximal F-torus T_H of H. Let γ_H be the element of $T_H(F)$ corresponding to γ under this isomorphism. The term $\Delta_{\mathrm{III}}(\gamma_H, \delta)$ in the absolute transfer factor is obtained by pairing the element $\mathrm{inv}(\gamma, \delta) \in H^1(F, T \xrightarrow{1-\theta} T)$ with the following element $\mathbf{A}$ in the dual hypercohomology group $H^1(W_F, \widehat{T} \xrightarrow{1-\widehat{\theta}} \widehat{T})$.

Assume for simplicity that $\mathcal{H} = {}^L H$. Using χ-data we embed ${}^L T_H$ in ${}^L H$, and then we compose this with the embedding of ${}^L H$ in ${}^L G$ that is part of our endoscopic data, obtaining an embedding ξ_{T_H} of ${}^L T_H$ in ${}^L G$. Let ${}^L G^1$ denote the subgroup of ${}^L G$ given as the semidirect product of the Weil group W_F and the identity component of the group of fixed points of $\widehat{\theta}$ on $\widehat{G}$. Note that ${}^L G^1$ is the L-group of a twisted endoscopic group G^1 of G. Again using our χ-data, we embed ${}^L T_H$ in ${}^L G^1$, and then we compose this with the canonical inclusion ${}^L G^1 \hookrightarrow {}^L G$, obtaining another embedding ξ_1 of ${}^L T_H$ in ${}^L G$. Replacing ξ_{T_H} by a conjugate under $\widehat{G}$ we may assume that ξ_{T_H} and ξ_1 agree on $\widehat{T}_H$. Then the difference between ξ_{T_H} and ξ_1 is measured by a 1-cocycle A of W_F in $\widehat{T}$, and $(1 - \widehat{\theta})(A^{-1})$ is the coboundary of an element $s_T \in \widehat{T}$ coming from the element s appearing in our endoscopic data. The class of the hypercocycle (A^{-1}, s_T) is the desired element $\mathbf{A}$ in $H^1(W_F, \widehat{T} \xrightarrow{1-\widehat{\theta}} \widehat{T})$.

Now we turn to our global results. In **[L2]** Langlands stabilized the elliptic regular terms in the trace formula; our second main goal in this paper is to do the same for the twisted trace formula (see **[R]** for the case of quadratic base change for unitary groups in three variables). Although the stabilization process is not difficult, it is surprisingly lengthy, partly because of the generality of the situation we consider. To ease the reader's task we will now summarize the main steps in the process.

Let F be a number field and G a connected reductive group over F. To make this introduction a little simpler we will assume that the center $Z(G)$ of G contains no non-trivial split torus. Let θ be an automorphism of G over F, and let $\mathbf{a}$ be an element of

$$H^1(W_F, Z(\widehat{G}))/\ker^1(W_F, Z(\widehat{G})).$$

Note that $\mathbf{a}$ determines a quasicharacter ω on $G(\mathbb{A})$, trivial on $G(F)$. The construction of ω from $\mathbf{a}$ is due to Langlands, but in this paper we find it convenient to use Borovoi's method **[Bo]** instead (see the proof of Theorem 5.1.D(2) for a review of Borovoi's method). We assume that ω is unitary and trivial on $Z(G)^\theta(\mathbb{A})$. Consider the Hilbert space

$$L^2 := L^2(G(F)\backslash G(\mathbb{A}))$$

and let

$$f \in C_c^\infty(G(\mathbb{A})).$$

Then f gives us a convolution operator $R(f)$ on L^2. Moreover θ and ω give us unitary operators $R(\theta)$, $R(\omega)$ on L^2; $R(\theta)$ is given by composition with θ^{-1} and $R(\omega)$ is given by pointwise multiplication by ω. The composition

$$R(f)R(\theta)R(\omega)$$

is an integral operator with kernel

$$K(h,g) = \omega(g) \sum_{\delta \in G(F)} f(h^{-1}\delta\theta(g)).$$

Let $\delta \in G(F)$ be θ-semisimple and strongly θ-regular. Write I_δ for the θ-centralizer $\mathrm{Cent}_\theta(\delta, G)$ of δ. As in (3.3) we denote by T_δ the centralizer in G of I_δ^0; then T_δ is a maximal torus of G preserved by $\mathrm{Int}(\delta) \circ \theta$ and I_δ coincides with the fixed points of $\mathrm{Int}(\delta) \circ \theta$ on T_δ. We say that δ is *θ-elliptic* if the identity component of

$$I_\delta / Z(G)^\theta$$

is anisotropic over F.

Denote by $G(F)_e$ the set of $\delta \in G(F)$ that are θ-semisimple, strongly θ-regular and θ-elliptic. Denote by $K_e(h,g)$ the corresponding part of the kernel $K(h,g)$:

$$K_e(h,g) := \omega(g) \sum_{\delta \in G(F)_e} f(h^{-1}\delta\theta(g)).$$

We are interested in the part of the twisted trace formula coming from $G(F)_e$, namely

$$T_e(f) := \int_{G(F)\backslash G(\mathbb{A})} K_e(g,g)\, dg/dx,$$

where dg is the Tamagawa measure on $G(\mathbb{A})$ (which is used to form the convolution operator $R(f)$) and dx is the counting measure on $G(F)$. As usual we can rewrite $T_e(f)$ as a sum of twisted orbital integrals (see (6.1.1))

$$T_e(f) = \sum_{\delta \in \Delta} c_G \cdot c_\delta \cdot \tau(I_\delta) \cdot O_{\delta\theta}(f). \tag{1}$$

Here $\tau(I_\delta)$ denotes the Tamagawa number of the diagonalizable group I_δ, and $O_{\delta\theta}(f)$ denotes the twisted orbital integral

$$\int_{I_\delta(\mathbb{A})\backslash G(\mathbb{A})} \omega(g) f(g^{-1}\delta\theta(g))\, dg/dt.$$

The numbers c_G and c_δ are defined in (6.1) (note that c_G is 1 since we assumed that $Z(G)^0$ is anisotropic). The sum is taken over a set Δ of representatives for the θ-conjugacy classes of elements $\delta \in G(F)_e$ such that ω is trivial on $I_\delta(\mathbb{A})$.

The next step (see (6.2)) is to rewrite (1) by combining the terms indexed by δ,δ' whenever δ,δ' are θ-conjugate under $G(\mathbb{A})$. Fix an element $\delta \in \Delta$. The set of $\delta' \in \Delta$ such that δ' is θ-conjugate to δ under $G(\mathbb{A})$ is in natural bijection with a certain finite