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ON LANGLANDS FUNCTORIALITY FROM CLASSICAL GROUPS TO GL_n

by

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Abstract. — This article is a survey of the descent method of Ginzburg, Rallis and Soudry. This method constructs, for an irreducible, automorphic, cuspidal, self-conjugate representation τ on $\mathrm{GL}_n(\mathbb{A})$, an irreducible, automorphic, cuspidal, generic representation $\sigma(\tau)$, on a corresponding quasi-split classical group G, which lifts weakly to τ. This construction works well also for all representations of $\mathrm{GL}_n(\mathbb{A})$, which are in the so called "tempered" part of the expected image of Langlands functorial lift from G to GL_n.

Résumé **(Sur la fonctorialité de Langlands des groupes classiques à GL_n).** — Cet article est une exposition de la méthode de descente de Ginzburg, Rallis et Soudry. Cette méthode construit, pour une représentation irréductible, automorphe et cuspidale τ telle que $\tau = \tau^*$, une représentation irréductible, automorphe, cuspidale et générique $\sigma(\tau)$ d'un groupe classique quasi-deployé G (qui dépend de GL_n et τ), telle que τ corresponde à $\sigma(\tau)$ par la correspondance fonctorielle faible (« weak lifting »). Cette construction est valable aussi pour toutes les représentations de $\mathrm{GL}_n(\mathbb{A})$ qui appartiennent à la partie dite « tempérée » de l'image de la correspondance fonctorielle de Langlands de G à GL_n.

Introduction

In these notes, I survey a long term work, joint with D. Ginzburg and S. Rallis, where we develop a descent method, which associates to a given irreducible automorphic representation τ of $\mathrm{GL}_n(\mathbb{A})$, an irreducible, automorphic, cuspidal, generic representation σ_τ on a given appropriate split classical group G, such that σ_ν lifts to τ_ν, for almost all places ν, where τ_ν is unramified. Of course, not every τ is obtained in such a way. We have to restrict ourselves to τ which lies in the expected

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(conjectural) image of the functorial lift from G to GL_n, restricted to cuspidal representations σ of $G(\mathbb{A})$. We restrict ourselves even more and consider only generic σ. This also applies to quasi-split unitary groups G. Here $\mathbb{A}$ denotes the adele ring of a number field F. Thus, for example, let E be a quadratic extension of F, and let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_E)$, such that its partial Asai L-function $L^S(\tau, \mathrm{Asai}, s)$ has a pole at $s = 1$. Then we construct an irreducible, automorphic, cuspidal, generic representation σ_τ of $U_{2n+1}(\mathbb{A})$, which lifts weakly (*i.e.* lifts at all places, where τ is unramified) to τ. Here, U_{2n+1} is the quasi-split unitary group in $2n+1$ variables, which corresponds to E. We regard it as an algebraic group over F. Note that σ_τ would probably be a generic member of "an L-packet which lifts to τ". Of course, σ_τ is a generic member of the near equivalence class which lifts to τ.

The basic ideas of our descent method (backward lift) can be found in [**GRS7, GRS8**]. A more detailed account appears in [**GRS1**], where we also start focusing on the descent from cuspidal τ on $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and $L(\tau, 1/2) \neq 0$, to ψ-generic cuspidal representations σ on the metaplectic cover of Sp_{2n}. We complete the study of this case (for non-cuspidal τ as well) in [**GRS2, GRS3, GRS4, GRS6**]. In [**GRS9**], we consider the lift from (split) SO_{2n+1} to GL_{2n}. I review this last case in Chapter 1 of these notes. Here we can prove more; namely, that the generic cuspidal representation σ_τ is unique up to isomorphism. This is achieved due to a "local converse theorem" for generic representations of $\mathrm{SO}_{2n+1}(k)$, over a p-adic field k, proved in [**Ji.So.1**]. In Chapter 2, I review integral representations for standard L-functions for $G \times \mathrm{GL}_m$ (valid only for generic representations). The integrals are of Rankin-Selberg or Shimura type. They are certain Gelfand-Graev, or Fourier-Jacobi coefficients applied to Eisenstein series or cusp forms. In Chapter 3, I review the descent from GL_n to G in general, and in Chapter 4, I illustrate various proofs through low rank examples.

This survey is the content of a minicourse that I gave at Centre Émile Borel, IHP, Paris, when I took part in the special semester in automorphic forms (Spring 2000). I thank the organizers H. Carayol, M. Harris, J. Tilouine, and M.-F. Vignéras for their invitation, and I thank my audience for their attention.

Frequently used notation

F – a number field.

$\mathbb{A} = \mathbb{A}_F$ – the adele ring of F.

F_ν – the completion of F at a place ν.

$\mathcal{O}_\nu$ – the ring of integers of F_ν, in case $\nu < \infty$.

$\mathcal{P}_\nu$ – the prime ideal of $\mathcal{O}_\nu$.

$q_\nu = |\mathcal{O}_\nu / \mathcal{P}_\nu|$.

$\mathrm{SO}_m(F) = \{g \in \mathrm{GL}_m(F) | {}^t g J g = J\}$, where $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

Let $\mathbb{R}^+$ denote the group of positive real numbers. Let $i : \mathbb{R}^+ \to \mathbb{A}^*$ be defined by $i(r) = \{x_\nu\}$, where for all finite places ν, $x_\nu = 1$, and for each archimedean place ν, $x_\nu = r$. We denote $i(\mathbb{R}^+) = \mathbb{A}_\infty^+$. For an irreducible representation τ, ω_τ denotes its central character. Sometimes we denote by V_τ a vector space realization of τ. When τ is an automorphic cuspidal representation, we assume that τ comes together with a specific vector space realization of cusp forms, which we sometimes denote by τ as well. Finally, given representations $\tau_1, \dots, \tau_r$ of $\mathrm{GL}_{n_1}(F_\nu), \dots, \mathrm{GL}_{n_r}(F_\nu)$ respectively, we denote by $\tau_1 \times \cdots \times \tau_n$ the representation of $\mathrm{GL}_n(F_\nu)$, $n = n_1 + \cdots + n_r$, induced from the standard parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_{n_1}(F_\nu) \times \cdots \times \mathrm{GL}_{n_r}(F_\nu)$, and the representation $\tau_1 \otimes \cdots \otimes \tau_r$.

1. The weak lift from SO_{2n+1} to GL_{2n}

In this chapter we survey the results on the weak lift from SO_{2n+1} to GL_{2n}, obtained after applying our descent method (backward lift). Together with the existence of this weak lift for generic representations [**C.K.PS.S.**], we obtain a fairly nice description of this weak lift, which turns out to be not weak at all.

1.1. Some preliminaries. — Let $\sigma \cong \otimes\sigma_\nu$ be an irreducible, automorphic, cuspidal representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. For almost all ν, σ_ν is unramified and is completely determined by a semisimple conjugacy class $[a_\nu]$ in ${}^L\mathrm{SO}_{2n+1}^\circ = \mathrm{Sp}_{2n}(\mathbb{C})$, so that $L(\sigma_\nu, s) = \det(I_{2n} - q_\nu^{-s} a_\nu)^{-1}$. Let i be the embedding $\mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C})$. Then the conjugacy class $[i(a_\nu)]$ in $\mathrm{GL}_{2n}(\mathbb{C})$ determines an unramified representation τ_ν of $\mathrm{GL}_{2n}(F_\nu)$, such that $L(\tau_\nu, s) = L(\sigma_\nu, s)$. The unramified representation τ_ν is called the local Langlands lift of σ_ν. This notion (of local Langlands lift) is conjecturally defined at all finite places and is well defined at archimedean places. For an archimedean place ν, σ_ν is determined by its Langlands parameter, which is an admissible homomorphism $\varphi_\nu : W_\nu \to \mathrm{Sp}_{2n}(\mathbb{C})$ from the Weil group of F_ν. The local lift of σ_ν is the representation τ_ν of $\mathrm{GL}_{2n}(F_\nu)$, whose Langlands parameter is $i \circ \varphi_\nu : W_\nu \to \mathrm{GL}_{2n}(\mathbb{C})$. (For finite places ν, where σ_ν is not unramified, σ_ν is conjecturally parameterized by an admissible homomorphism from the Weil-Deligne group $\varphi_\nu : W_\nu \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$, and an irreducible representation τ_ν of $\mathrm{GL}_{2n}(F_\nu)$ would be a local lift of σ_ν, if τ_ν corresponds to the homomorphism $i \circ \varphi_\nu$, under the local Langlands reciprocity law for GL_{2n}, now proved by Harris-Taylor [**H.T.**] and by Henniart [**H**].) An irreducible, automorphic representation $\tau \cong \otimes\tau_\nu$ is a weak lift of σ, if for every archimedean place ν and for almost all finite places ν where σ_ν is unramified, τ_ν is the local lift of σ_ν. Using the converse theorem for GL_m [**C.PS.**] and L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ constructed and studied by Shahidi [**Sh1**], the existence of a weak lift from SO_{2n+1} to GL_{2n} was established for *globally generic* σ, by J. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi.

Theorem **([C.K.PS.S.])**. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$. *Then σ has a weak lift to* $\mathrm{GL}_{2n}(\mathbb{A})$.

Here we remark that a weak lift of σ is realized as an irreducible subquotient of the space of automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$. Moreover, by the strong multiplicity one property for GL_{2n} [**J.S.**], all weak lifts of σ are constituents of one representation of $\mathrm{GL}_{2n}(\mathbb{A})$ of the form $\tau_1 \times \cdots \times \tau_r$, where τ_i are (irreducible, automorphic) cuspidal representations of $\mathrm{GL}_{m_i}(\mathbb{A})$, $m_1 + \cdots + m_r = 2n$ and the set $\{\tau_1, \ldots, \tau_r\}$ is uniquely determined. In particular, if σ has a cuspidal weak lift, then it is unique. We are going to describe the image of the above weak lift, starting with its cuspidal part.

1.2. The cuspidal part of the image. — Let σ be an irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Assume that σ has a cuspidal weak lift τ on $\mathrm{GL}_{2n}(\mathbb{A})$. As we just remarked, τ is uniquely determined (even with multiplicity one). Clearly $\tau_\nu \cong \widehat{\tau}_\nu$ (and $\omega_{\tau_\nu} = 1$), for almost all ν. By the strong multiplicity one and multiplicity one properties for GL_{2n}, [**J.S.**], [**Sk**], we have $\tau = \widehat{\tau}$, *i.e.* τ is self-dual. (Similarly, $\omega_\tau = 1$). Let S be a finite set of places, including those at infinity, outside which σ and τ are unramified. We have

$$L^S(\sigma \times \tau, s) = L^S(\tau \times \tau, s) = L^S(\widehat{\tau} \times \tau, s),$$

and hence $L^S(\sigma \times \tau, s)$ has a pole at $s = 1$. Recall that

$$L^S(\tau \times \tau, s) = L^S(\tau, \mathrm{sym}^2, s) L^S(\tau, \Lambda^2, s).$$

By Langlands' conjectures, one expects τ to be "symplectic", and so the pole of $L^S(\tau \times \tau, s)$ at $s = 1$ should come from $L^S(\tau, \Lambda^2, s)$.

Theorem 1. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$. *Assume that σ has a cuspidal weak lift τ on* $\mathrm{GL}_{2n}(\mathbb{A})$. *Then* $L^S(\tau, \Lambda^2, s)$ *has a pole at* $s = 1$.

Proof. — Let us express the pole at $s = 1$ of $L^S(\sigma \times \tau, s)$ through a Rankin-Selberg type integral which represents this L-function [**So1**], [**G.PS.R.**]. It has the form

$$\mathcal{L}(\varphi_\sigma, f_{\tau,s}) = \int_{\mathrm{SO}_{2n+1}(F)\backslash \mathrm{SO}_{2n+1}(\mathbb{A})} \varphi_\sigma(g) E^\psi(f_{\tau,s}, g) dg, \tag{1.1}$$

where φ_σ is a cusp form in the space of σ, $E(f_{\tau,s}, \cdot)$ is an Eisenstein series on split $\mathrm{SO}_{4n}(\mathbb{A})$ corresponding to a K-finite holomorphic section $f_{\tau,s}$ in $\mathrm{Ind}_{P_{2n}(\mathbb{A})}^{\mathrm{SO}_{4n}(\mathbb{A})} \tau|\det\cdot|^{s-1/2}$, where P_{2n} is the Siegel parabolic subgroup of SO_{4n}. E^ψ denotes a Fourier coefficient along the subgroup

$$N_n = \left\{ u = \begin{pmatrix} z & y & e \\ & I_{2n+2} & y' \\ & & z^* \end{pmatrix} \in \mathrm{SO}_{4n} \;\middle|\; z \in Z_{n-1} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\},$$