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SUN-YUNG ALICE CHANG
PAUL C. YANG

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THE Q-CURVATURE EQUATION IN CONFORMAL GEOMETRY

by

Sun-Yung Alice Chang & Paul C. Yang

Dedicated to J.P. Bourguignon on his 60^{th} birthday

Abstract. — In this paper we survey some analytic results concerned with the top order Q-curvature equation in conformal geometry. Q-curvature is the natural generalization of the Gauss curvature to even dimensional manifolds. Its close relation to the Pfaffian, the integrand in the Gauss-Bonnet formula, provides a direct relation between curvature and topology.

Résumé **(L'équation de Q-courbure en géométrie conforme).** — Dans cet article nous examinons certains résultats analytiques autour de l'équation de Q-courbure d'ordre maximal en géométrie conforme. La Q-courbure est la généralisation naturelle de la courbure de Gauss aux variétés de dimension paire. Sa proximité avec le pfaffien (l'intégrande de la formule de Gauss-Bonnet) nous fournit une relation directe entre géométrie et topologie.

1. Introduction

Recently, there is a lot of interest in the study of higher order Q-curvature invariant. This notion arises naturally in conformal geometry in the context of conformally covariant operators. Paneitz ([**23**], see also [**6**]) gave the first construction of the fourth order conformally covariant Paneitz operator in the context of Lorentzian geometry in dimension four. Based on the ambient metric construction introduced by Fefferman and Graham ([**14**],[**15**]), Graham-Jenne-Mason and Sparling [**18**] systematically constructed conformally covariant operators of higher orders. Each such operator gives rise to a semi-linear elliptic equation analogous to the Yamabe equations which

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we shall call the Q-curvature equation. These equations share a number of common features. Among these we mention the following:

(i) the lack of compactness: the nonlinearity always occur at the critical exponent, for which the Sobolev embedding is not compact;

(ii) the lack of maximum principle: for example, it is not known whether the solution of the fourth order Q-curvature equation on manifolds of dimensions greater than four may touch zero.

In spite of these difficulty, there has been significant progress on questions of existence, regularity and classification of entire solutions for these equations in the recent work of Djadli-Malchiodi [**13**], Adimurthi-Robert-Struwe [**1**] and X.Xu [**25**]. On the other hand, in the case when the dimension is even $n = 2k$, the Branson-Paneitz operator and its associated Q-curvature equation is more accessible. In this article, we will give a brief survey of two results for the Q-curvature equation, each of which makes use of its close relation to the Pfaffian; both of these results are joint works with Jie Qing. The first [**10**] is a generalization of the Cohn-Vossen-Huber inequality ([**22**]) to complete conformal metrics on domains in $\mathbb{R}^4$. The second gives a Gauss-Bonnet type formula for Poincaré-Einstein metrics in which the renormalized volume plays a role. As the original article [**12**] of the second result appeared in Russian, we provide an exposition with some details. In section two, we review the notion of conformally covariant equations, their associated Q-curvatures and the associated boundary operators for manifold with boundary. We then provide an outline for these two results in sections three to five.

2. Conformally covariant operators and the Q-curvature equation

In general, we call a metrically defined operator A defind on a Riemannian manifold (M^n, g) conformally covariant of bidegree (a, b), if under the conformal change of metric $g_w = e^{2w}g$, the pair of corresponding operators A_w and A are related by

$$A_w(\varphi) = e^{-bw}A(e^{aw}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n).$$

A basic example is the conformal Laplacian $L \equiv -\Delta + \frac{n-2}{4(n-1)}R$ where R is the scalar curvature of the metric. The conformal Laplacian is conformally covariant of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$, and the associated curvature equation is the equation for prescribing scalar curvature: writing $e^w = u^{\frac{2}{n-2}}$ we have

$$Lu = \frac{n-2}{4(n-1)}R_u u^{\frac{n+2}{n-2}}, \tag{1}$$

where R_u is the scalar curvature of the metric $g_w = g^{2w}g = u^{\frac{4}{n-2}}g$. In case of surfaces, the corresponding Q-curvature equation becomes the equation for prescribing Gauss curvature:

$$- \Delta w + K = K_w e^{2w}, \tag{2}$$

where K_w is the Gaussian curvature for the metric g_w, and we have the Gauss-Bonnet formula:

$$2\pi\chi(M) = \int_M K dA. \tag{3}$$

In dimension four, S. Paneitz found the fourth order conformally covariant operator:

$$P_4\varphi = P\varphi \equiv \Delta^2\varphi + \delta\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)d\varphi\right] \tag{4}$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor.

For example:

- On $(R^4, |dx|^2)$, $P = \Delta^2$,
- On (S^4, g_c), $P = \Delta^2 - 2\Delta$,
- On (M^4, g), g Einstein, $P = (-\Delta) \circ (L)$.

The Paneitz operator P has bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\phi) = e^{-4\omega} P_g(\phi) \ \ \forall \phi \in \mathcal{C}^\infty(M^4)\,. \tag{5}$$

The fourth order Q-curvature is given by

$$Q \ = \ \frac{1}{6}(-\Delta R + R^2 - 3|\mathrm{Ric}|^2). \tag{6}$$

Under the conformal change of metric $g_w \ = \ e^{2w}g$, the Q-curvature equation (see [**6**], also [**8**]) takes the form

$$Pw + Q \ = \ Q_w e^{4w}, \tag{7}$$

where Q_w is the Q curvature for the metric g_w.

The Gauss-Bonnet formula in dimension four may be written as

$$8\pi^2\chi(M) = \int_M (|W|^2 + Q)dV, \tag{8}$$

where W is the Weyl tensor. Since $|W_g|_g = e^{-2w}|W_{g_w}|_{g_w}$, on manifold of dimension four, $|W|^2\, dV$ is a pointwise conformal invariant, thus it follows from the Gauss-Bonnet formula that the Q-curvature integral is a global conformal invariant.

For 4-manifold X^4 with boundary M^3 and a Riemannian metric g defined on closure of X^4, Chang-Qing [**9**] derived the matching boundary operator

$$P_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta - \tilde{\Delta}\frac{\partial}{\partial n} - \frac{2}{3}H\tilde{\Delta} + L_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + \left(\frac{1}{3}R - R_{\alpha N\alpha N}\right)\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla}H\cdot\tilde{\nabla}. \tag{9}$$

with the associated third order curvature invariant

$$T = \frac{1}{12}\frac{\partial}{\partial n}R + \frac{1}{6}RH - R_{\alpha N\beta N}L_{\alpha\beta} + \frac{1}{9}H^3 - \frac{1}{3}TrL^3 - \frac{1}{3}\tilde{\Delta}H, \tag{10}$$

where where $\frac{\partial}{\partial n}$ is the outer normal derivative, $\tilde{\Delta}$ is the trace of the Hessian of the metric on the boundary, $\tilde{\nabla}$ is the derivative in the boundary, L is the second fundamental form of boundary, $H = \mathrm{Tr}L$, N denotes the inner normal direction. We have used an orthonormal frame and let the latin indices run through the ambient indices and the Greek indices only run through the boundary directions, and all curvature are taken with respect to the metric g.

In particular, via the conformal change of metrics $g_w = e^{2w}g$, P_3 and T satisfy the equation:

$$P_3 w + T = T_w e^{3w} \quad \text{on} \quad M, \tag{11}$$

and

$$(P_3)_w = e^{-3w} P_3 \quad on \quad M. \tag{12}$$

The Chern-Gauss-Bonnet formula for 4-manifolds with boundary is then modified with a boundary term:

$$8\pi^2 \chi(X) = \int_X (|W|^2 + Q)dv + 2\oint_M (T - \mathcal{L}_4 - \mathcal{L}_5)d\sigma. \tag{13}$$

In the boundary integral above the invariants $\mathcal{L}_4$ and $\mathcal{L}_5$ involve the ambient curvature tensor and the second fundamental form L_{ab}, and their expressions are

$$\mathcal{L}_4 = -\frac{RH}{3} + R_{\alpha N \alpha N} H - R_{\alpha N \beta N} L_{\alpha\beta} + R_{\gamma\alpha\gamma\beta} L_{\alpha\beta},$$

and

$$\mathcal{L}_5 = -\frac{2}{9} L_{\alpha\alpha} L_{\beta\beta} L_{\gamma\gamma} + L_{\alpha\alpha} L_{\beta\gamma} L_{\beta\gamma} - L_{\alpha\beta} L_{\beta\gamma} L_{\gamma\alpha}.$$

Analogous to the Weyl term, $\mathcal{L}_4$ and $\mathcal{L}_5$ are boundary invariant of order three which are pointwise invariant under conformal change of metrics. Hence

$$\int_X Q dv \; + \; 2\oint_M T d\sigma \tag{14}$$

is a global conformal invariant.

In dimension four, an important result is the following criteria for positivity of the Paneitz operator due to Gursky-Viaclovsky:

Theorem 1 **([21])**. — *Let (M^4, g) be a metric with positive Yamabe constant $Y(M,g) = inf_{u \neq 0} \frac{\int Lu \cdot u}{||u||_4^2}$, and satisfying*

$$\int_M Q dv + \frac{1}{6}(Y(M,g))^2 \geq 0,$$

then the Paneitz operator is positive except for constants.

It is an open question whether there is an analogous result in higher dimensions.