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## *Local theta correspondences between supercuspidal representations*

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# LOCAL THETA CORRESPONDENCES BETWEEN SUPERCUSPIDAL REPRESENTATIONS

BY Hung Yean LOKE AND Jia-Jun MA

Abstract. – By the works of Yu, Kim and Hakim-Murnaghan, we have a parameterization and construction of all supercuspidal representations of a reductive $p$-adic group in terms of supercuspidal data, when $p$ is sufficiently large. In this paper, we will define a correspondence of supercuspidal data via moment maps and theta correspondences over finite fields. Then we will show that local theta correspondences between supercuspidal representations are completely described by this notion. In Appendix B, we give a short proof of a result of Pan on "depth preservation".

Résumé. – Par les travaux de Yu, Kim et Hakim-Murnaghan, on a une paramétrisation et une construction de toutes les représentations supercuspidales d'un groupe réductif $p$-adique en termes de données supercuspidales, quand $p$ est suffisamment grand. Dans cet article, nous définirons une correspondance entre les données supercuspidales par l'intermédiaire d'applications moments et de correspondances thêta sur des corps finis. Ensuite, nous montrerons que les correspondances thêta locales entre les représentations supercuspidales sont complètement décrites par cette notion. Dans l'Appendice B, nous fournissons une courte démonstration d'un résultat de Pan sur la « préservation de la profondeur ».

## 1. Introduction

In this paper, we give an explicit description of the local theta correspondences between tamely ramified supercuspidal representations in terms of the supercuspidal data developed in [15, 37, 17, 11].

### 1.1. Notation

Throughout this paper, we fix a non-Archimedean local field $F$ of characteristic zero with ring of integers $\mathfrak{o}$, and finite residual field $\mathfrak{f}$. Let "val" denote the normalized valuation map such that $\mathrm{val}(F) = \mathbb{Z}$. Suppose $E$ is a finite extension of $F$ or the central simple quaternion division algebra over $F$, let $\mathfrak{o}_E$ denote its ring of integers, let $\mathfrak{p}_E$ denote the maximal ideal in $\mathfrak{o}_E$ and let $\mathfrak{f}_E := \mathfrak{o}_E/\mathfrak{p}_E$ denote the residue field. We continue to let "val" denote the natural extension of valuations to $E$. When $E = F$, we sometimes omit the subscript. We


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fix a non-trivial additive character $\psi\colon F \to \mathbb{C}^\times$ with conductor $\mathfrak{p}$ (i.e., $\psi|_{\mathfrak{p}}$ is trivial but $\psi|_{\mathfrak{o}}$ is non-trivial). Let $\overline{\psi}$ denote the additive character on $\mathfrak{f}$ induced by $\psi$. For a vector space $\mathfrak{V}$ with an endomorphism $\star$, we let $\mathfrak{V}^{\star,\varepsilon}$ denote the $\varepsilon$-eigenspace of $\star$ in $\mathfrak{V}$.

## 1.2. The set of data

Let $(D,\tau)$ denote one of the division algebras over $F$ given in Section 2.1 with an $F$-linear involution $\tau$. Let $\epsilon \in \{\pm 1\}$ and $\epsilon' = -\epsilon$. Let $(V, \langle\ ,\ \rangle_V)$ (respectively $(V', \langle\ ,\ \rangle_{V'})$) denote a right $D$-module equipped with an $\epsilon$-Hermitian form $\langle\ ,\ \rangle_V$ (respectively $\epsilon'$-Hermitian form $\langle\ ,\ \rangle_{V'}$). Then $W := V \otimes_D V'$ is naturally a symplectic space. Let $(G,G') = (\mathrm{U}(V),\mathrm{U}(V'))$ be an irreducible type I reductive dual pair in the symplectic group $\mathrm{Sp} := \mathrm{Sp}(W)$. For any subset $E$ of Sp let $\widetilde{E}$ be its inverse image in the metaplectic $\mathbb{C}^\times$-cover $\widetilde{\mathrm{Sp}}(W)$ of $\mathrm{Sp}(W)$. See Section 2 for more details of the notation.

We assume that $p$ is large enough compared to the sizes of $G$ and $G'$ since we need the hypotheses in [17, §3.5] to hold. We will give a lower bound for $p$ in Corollary 3.2. We will review the construction of supercuspidal representations for $\widetilde{G}$ following [37, 17] in Section 3. Let $\Sigma := (x,\Gamma,\phi,\rho)$ be a supercuspidal datum as in [17]. We briefly explain the entries in $\Sigma$: (i) $\Gamma$ is a semisimple element in $\mathfrak{g}$ and $G^0 := Z_G(\Gamma)$; (ii) $x$ is a point in the building $\mathscr{B}(G^0)$ of $G^0$; (iii) $\phi$ and $\rho$ are certain representations of $G^0_x$. See Definition 3.4 for details. Then $\Sigma$ will determine an open compact subgroup $K \subseteq G$ and an irreducible $K$-module $\eta_\Sigma$ and, $\pi_\Sigma := \text{c-Ind}_K^G \eta_\Sigma$ is a supercuspidal representation of $G$. By [17], under the assumption that $p$ is large enough, this construction gives all supercuspidal representations of $G$. Let $\mathscr{D}_V$ be the set of all supercuspidal data and let $\hat{G}_{\mathrm{sc}}$ be the equivalence classes of irreducible supercuspidal $G$-modules. In [11] an equivalence relation $\sim$ on $\mathscr{D}_V$ is defined so that $\overline{\mathscr{D}}_V := \mathscr{D}_V/\sim\, \to \hat{G}_{\mathrm{sc}}$ given by $[\Sigma] \mapsto [\pi_\Sigma]$ is a bijection. In other words, $\overline{\mathscr{D}}_V$ parametrizes $\hat{G}_{\mathrm{sc}}$. In fact, the equivalence relation is just $G$-conjugacy in our situation (cf. Definition 3.6).

Now we consider the covering group $\widetilde{G}$. It is well known that the cover $\widetilde{K} \twoheadrightarrow K$ splits. Given a certain splitting $\xi\colon K \to \widetilde{K}$, we identify $\widetilde{K}$ with $K \times \mathbb{C}^\times$. We call $\widetilde{\Sigma} := (\Sigma,\xi) = (x,\Gamma,\phi,\rho,\xi)$ a supercuspidal datum of $\widetilde{G}$. Define $\widetilde{\eta}_{\widetilde{\Sigma}} := \eta_\Sigma \boxtimes \mathrm{id}_{\mathbb{C}^\times}$ which is an irreducible $\widetilde{K}$-module. Then $\tilde{\pi}_{\widetilde{\Sigma}} := \text{c-Ind}_{\widetilde{K}}^{\widetilde{G}} \widetilde{\eta}_{\widetilde{\Sigma}}$ is an irreducible supercuspidal representation of $\widetilde{G}$. We will see in Section 3.5.4 that under the assumption that $p$ is large enough, the construction of $\tilde{\pi}_{\widetilde{\Sigma}}$ exhausts all the irreducible supercuspidal genuine[(1)] representations of $\widetilde{G}$. The equivalence relation on the set of data of $\widetilde{G}$ could also be deduced from that of $G$ easily (cf. Section 3.5).

## 1.3. Statement of the main theorem

We retain the notation in Section 1.2. Fix a Witt tower $\mathscr{T}'$ of $\epsilon'$-Hermitian spaces. The covering group $\widetilde{G}$ in the dual pair $(G,G') = (\mathrm{U}(V),\mathrm{U}(V'))$ for all $V' \in \mathscr{T}'$ are canonically isomorphic to one another. Let $\omega$ be the Weil representation of $\widetilde{\mathrm{Sp}}(W)$ with respect to the character $\psi$ and let

$$\mathscr{R}(\widetilde{G},\omega) := \{\, \tilde{\pi} \in \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}) \mid \mathrm{Hom}_{\widetilde{G}}(\omega,\tilde{\pi}) \neq 0 \,\} \tag{1.1}$$

(1) Here genuine means $\mathbb{C}^\times \subseteq \widetilde{G}$ acts by multiplication.

be the equivalence classes of irreducible smooth genuine $\widetilde{G}$-modules which could be realized as a quotient of $\omega$. Let $\theta_{V,V'}\colon \mathscr{R}(\widetilde{G},\omega) \to \mathscr{R}(\widetilde{G}',\omega)$ denote the theta correspondence map.

Let $\tilde{\pi}$ be an irreducible supercuspidal genuine $\widetilde{G}$-module. Note that the $\tilde{\pi}$-isotypic component $\omega[\tilde{\pi}]$ of $\omega$ is naturally a $\widetilde{G}\times\widetilde{G}'$ module, say $\omega[\tilde{\pi}] \cong \tilde{\pi} \boxtimes \Theta_{V,V'}(\tilde{\pi})$ where $\Theta_{V,V'}(\tilde{\pi})$ is a genuine $\widetilde{G}'$-module. Let

$$m_{\mathscr{T}'}(\tilde{\pi}) = \min\{\, \dim_D(V'') \mid \Theta_{V,V''}(\tilde{\pi}) \neq 0 \text{ where } V'' \in \mathscr{T}'\,\}$$

which is called the *first occurrence index* of $\pi$ with respect to the Witt tower $\mathscr{T}'$.

It is well known that (cf. [23, Section 3.IV.4, théorème principal]):

(i) $\Theta_{V,V'}(\tilde{\pi})$ is either zero or irreducible.

(ii) $m_{\mathscr{T}'}(\tilde{\pi}) \le 2\dim V + a_{\mathscr{T}'}$ where $a_{\mathscr{T}'} = \min\{\, \dim_D V'' \mid V'' \in \mathscr{T}'\,\}$ is the dimension of the anisotropic kernel in $\mathscr{T}'$ (cf. [20]).

(iii) $\Theta_{V,V'}(\tilde{\pi}) \neq 0$ if and only if $\dim_D(V') \ge m_{\mathscr{T}'}(\tilde{\pi})$ in which case $\theta_{V,V'}(\tilde{\pi}) = \Theta_{V,V'}(\tilde{\pi})$.

(iv) $\theta_{V,V'}(\tilde{\pi})$ is supercuspidal if and only if $\dim(V') = m_{\mathscr{T}'}(\tilde{\pi})$. In this case, we say that the first occurrence of $\tilde{\pi}$ is at $V'$.

The aim of this paper is to describe the first occurrences of theta lifts of supercuspidal representations in terms of the supercuspidal data.

Let

$$\overline{\mathscr{D}}_{\mathscr{T}'} = \bigsqcup_{V'\in\mathscr{T}'} \overline{\mathscr{D}}_{V'}. \tag{1.2}$$

Using the moment maps and theta correspondences over finite fields, we will define theta lifts of equivalence classes of supercuspidal data in Section 5, i.e., we will define a map

$$\vartheta_{V,\mathscr{T}'}\colon \overline{\mathscr{D}}_V \hookrightarrow \overline{\mathscr{D}}_{\mathscr{T}'}. \tag{1.3}$$

Fix a pair of data $(\Sigma,\Sigma') \in \mathscr{D}_V \times \mathscr{D}_{V'}$. There is a canonical splitting

$$\xi_{x,x'}\colon K\times K' \longrightarrow \widetilde{K}\times\widetilde{K}'$$

constructed from the generalized lattice model (cf. (2.4)). We always set $\widetilde{\Sigma} = (\Sigma, \xi_{x,x'}|_K)$ and $\widetilde{\Sigma}' = (\Sigma', \xi_{x,x'}|_{K'})$.

Main Theorem. – *(i) Suppose* $\Sigma \in \mathscr{D}_V$ *and* $[\Sigma'] := \vartheta_{V,\mathscr{T}'}([\Sigma]) \in \overline{\mathscr{D}}_{V'}$ *for certain* $V' \in \mathscr{T}'$. *Then* $\theta_{V,V'}(\tilde{\pi}_{\widetilde{\Sigma}}) = \tilde{\pi}'_{\widetilde{\Sigma}'}$. *(ii) Conversely, suppose* $\theta_{V,V'}(\tilde{\pi}) = \tilde{\pi}'$, *such that* $\tilde{\pi}$ *and* $\tilde{\pi}'$ *are supercuspidal representations. Then there exists* $\Sigma \in \mathscr{D}_V$ *such that* $\tilde{\pi} = \tilde{\pi}_{\widetilde{\Sigma}}$ *and* $\tilde{\pi}' = \tilde{\pi}'_{\widetilde{\Sigma}'}$ *where* $[\Sigma'] = \vartheta_{V,\mathscr{T}'}([\Sigma])$ *and* $\mathscr{T}'$ *is the Witt class of* $V'$.

Remarks. – 1. If $\tilde{\pi}$ is a depth zero supercuspidal representation, then $\vartheta_{V,\mathscr{T}'}(\tilde{\pi})$ is essentially constructed in [28].

2. After the completion of the first draft of this paper, we received a preprint [30] from Pan which describes the theta lifts of certain positive depth supercuspidal representations.

3. The main theorem generalizes our earlier results with Savin for epipelagic representations [22].

4. The construction of $\vartheta_{V,\mathscr{T}'}$ provides a criterion on the occurrence of supercuspidal representations by conditions on the isomorphism classes of the Hermitian spaces modulo the theta correspondences over finite fields. On the other hand, for some supercuspidal