Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 140
Fascicule 4

2012

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 533-549

Bull. Soc. math. France
140 (4), 2012, p. 533–549

PULLING BACK COHOMOLOGY CLASSES AND DYNAMICAL DEGREES OF MONOMIAL MAPS

BY JAN-LI LIN

ABSTRACT. — We study the pullback maps on cohomology groups for equivariant rational maps (i.e., monomial maps) on toric varieties. Our method is based on the intersection theory on toric varieties. We use the method to determine the dynamical degrees of monomial maps and compute the degrees of the Cremona involution.

1. Introduction

For a meromorphic map $f : X \dashrightarrow X$ on a compact Kähler manifold X of dimension n, and for $1 \leq k \leq n$, there is a well-defined pullback map $f^* : H^{k,k}(X;\mathbb{R}) \to H^{k,k}(X;\mathbb{R})$. The *k-th dynamical degree* of f is then defined as

$$\lambda_k(f) = \lim_{\ell\to\infty} \left\|(f^\ell)^*\right\|^{1/\ell}.$$

The dynamical degrees $\lambda_1(f),\ldots,\lambda_n(f)$ form an important family of birational invariants. The computation and properties of the first dynamical degree of various maps is an active research topic. For example, see [1, 2, 3, 5, 9, 19] and the references therein. On the other hand, much less is known about the higher dynamical degrees, see [6, 20]. In this paper, we focus on monomial

Texte reçu le 9 novembre 2010, révisé et accepté le 3 février 2011.

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2010 Mathematics Subject Classification. — 37F10; 14M25.

Key words and phrases. — Dynamical degrees, topological entropy, monomial maps.

maps, and we compute the higher dynamical degrees of dominant monomial maps.

Given an $n \times n$ integer matrix $\psi = (a_{i,j})$, the associated monomial map $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is defined by

$$f(x_1, \dots, x_n) = (\prod_{j=1}^n x_j^{a_{1,j}}, \dots, \prod_{j=1}^n x_j^{a_{n,j}}).$$

The map f gives rise to an equivariant rational map on toric varieties which contain $(\mathbb{C}^*)^n$ as a dense open subset, and f is dominant if and only if $\det(\psi) \neq 0$. The main theorem of this paper is the following.

THEOREM 1. — *Let f be the monomial map induced by ψ, and let f be dominant. Let $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n|$ be the eigenvalues (counting multiplicities) of ψ. Then the k-th dynamical degree is*

$$\lambda_k(f) = |\mu_1 \cdots \mu_k|.$$

This answers Question 9.9 in [16]. As a corollary, we also obtain the following formula for the topological entropy of monomial maps, which answers Question 9.1 in [16].

COROLLARY. — *The topological entropy of a dominant monomial map f is*

$$h_{\rm top}(f) = \sum_{|\mu_i|>1} \log |\mu_i|.$$

The theory of toric varieties is a useful tool for understanding the dynamics of monomial maps, as was shown in the papers ([8, 17, 18]). Using the intersection theory on toric varieties, we are able to translate the computation of cohomology classes into much simpler computation involving cones and lattice points.

We note that C. Favre and E. Wulcan [10] have obtained the same results on dynamical degrees using related but different methods from ours.

In Section 2, we will review results from the intersection theory of toric varieties. Then a method to pull back cohomology classes by monomial maps on toric varieties is developed in Section 3. The method is applied in Section 4 to give a concrete formula for pulling back cohomology classes by monomial maps on $(\mathbb{P}^1)^n$. We use the formula to prove our main result about dynamical degrees. In the last section, we apply our method to compute the degrees of the Cremona involution J on $\mathbb{P}^n$.

I heartily thank my advisor Eric Bedford for his frequent conversation, encouragement and guidance. I would also like to thank Julie Déserti and Tuyen Trung Truong for helpful discussions. Finally, I want to thank Charles Favre for bringing the reference [14] to the my attention and the referee for various comments.

2. Intersection theory on toric varieties

In this section, we briefly review the intersection theory on toric varieties. Results are stated without proof. For more details, see [13].

2.1. Chow homology groups of toric varieties. — Following the notations of [11, 13], we let $X = X(\Delta)$ be a toric variety corresponding to a fan Δ in a lattice N of rank n. Each cone $\tau \in \Delta$ corresponds to an orbit O_τ of the torus action, and the closure of O_τ in $X(\Delta)$ is denoted by $V(\tau)$. Each $V(\tau)$ is invariant under the torus action. Conversely, every torus-invariant closed subvariety of X is of the form $V(\tau)$. The dimension of $V(\tau)$ is given by

$$\dim(V(\tau)) = n - \dim(\tau) = \operatorname{codim}(\tau).$$

For each cone τ, define N_τ to be the sublattice of N generated by $\tau \cap N$, and $M(\tau) := \tau^\perp \cap M$. Every nonzero $u \in M(\tau)$ determines a rational function χ^u on $V(\tau)$. The divisor of χ^u is given by

$$\operatorname{div}(\chi^u) = \sum_\sigma \langle u, n_{\sigma,\tau}\rangle \cdot V(\sigma), \tag{2.1}$$

where the sum is over all cones $\sigma \in \Delta$ which contain τ with $\dim(\sigma) = \dim(\tau)+1$, and $n_{\sigma,\tau}$ is a lattice point in σ whose image generates the 1-dimensional lattice N_σ/N_τ.

For an arbitrary variety X, we define the *Chow group* $A_k(X)$ as the group generated by the k-dimensional irreducible closed subvarieties of X, with relations generated by divisors of nonzero rational functions on some $(k+1)$-dimensional subvariety of X. In the case of toric varieties, there is a nice presentation of the Chow groups in terms of torus-invariant subvarieties and torus-invariant relations.

PROPOSITION 2.1. — (a) *The Chow group $A_k(X)$ of a toric variety X is generated by the classes $[V(\sigma)]$ where σ runs over all cones of codimension k in the fan Δ.*

(b) *The group of relations on these generators is generated by all relations (2.1), where τ runs over cones of codimension $k+1$ in Δ, and u runs over a generating set of $M(\tau)$.*

For the proof, see [13, Proposition 1.1].

2.2. Chow cohomology of toric varieties; Minkowski weights. — For a complete toric variety X, it is known ([13, Proposition 1.4]) that the Chow cohomology group $A^k(X)$ is canonically isomorphic to $\mathrm{Hom}(A_k(X), \mathbb{Z})$. Thus we can describe $A^k(X)$ as follows.

Let Δ be a complete fan in a lattice N, and let $\Delta^{(k)}$ denote the subset of all cones of codimension k.

Definition. — An integer-valued function c on $\Delta^{(k)}$ is called a *Minkowski weight* of codimension k if it satisfies the relations

$$\sum_{\sigma \in \Delta^{(k)},\ \sigma \supset \tau} \langle u, n_{\sigma,\tau} \rangle \cdot c(\sigma) = 0 \tag{2.2}$$

for all cones $\tau \in \Delta^{(k+1)}$ and all $u \in M(\tau)$.

Proposition 2.2. — *The Chow cohomology group $A^k(X)$ of a complete toric variety $X = X(\Delta)$ is canonically isomorphic to the group of Minkowski weights of codimension k on Δ.*

This proposition is an immediate consequence of the description of $A_k(X)$ in Proposition 2.1 and the isomorphism $A^k(X) \cong \mathrm{Hom}(A_k(X), \mathbb{Z})$.

2.3. Relations with the usual (co)homology groups. — For a nonsingular complete toric variety $X = X(\Delta)$, we have $H_k(X; \mathbb{Z}) = 0$ and $H^k(X; \mathbb{Z}) = 0$ for k odd (see [11, p.92]). Furthermore, $A_k(X) \cong H_{2k}(X; \mathbb{Z})$ and $A^k(X) \cong H^{2k}(X; \mathbb{Z})$ (see [11, p.102 and p.106]). This means

$$A_*(X) \cong H_*(X; \mathbb{Z}) \text{ and } A^*(X) \cong H^*(X; \mathbb{Z}),$$

and both isomorphisms double the degree.

Moreover, suppose X is also projective, thus Kähler. We have the Hodge decomposition $H^k(X; \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X)$. The following result is known for toric varieties ([4, Proposition 12.11]):

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X) = 0 \text{ for } p \neq q.$$

As a consequence, we have $A^k(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^{2k}(X; \mathbb{R}) \cong H^{k,k}(X; \mathbb{R})$.