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GLOBAL IN TIME STRICHARTZ INEQUALITIES ON ASYMPTOTICALLY FLAT MANIFOLDS WITH TEMPERATE TRAPPING

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GLOBAL IN TIME STRICHARTZ INEQUALITIES ON ASYMPTOTICALLY FLAT MANIFOLDS WITH TEMPERATE TRAPPING

Jean-Marc Bouclet, Haruya Mizutani

Abstract. – We prove global Strichartz inequalities for the Schrödinger equation on a large class of asymptotically conical manifolds. Letting P be the nonnegative Laplace operator and $f_0 \in C_0^\infty(\mathbb{R})$ be a smooth cutoff equal to 1 near zero, we show first that the low frequency part of any solution $e^{-itP}u_0$, i.e., $f_0(P)e^{-itP}u_0$, enjoys the same global Strichartz estimates as on $\mathbb{R}^n$ in dimension $n \geq 3$. We also show that the high energy part $(1-f_0)(P)e^{-itP}u_0$ also satisfies global Strichartz estimates without loss of derivatives outside a compact set, even if the manifold has trapped geodesics but in a temperate sense. We then show that the full solution $e^{-itP}u_0$ satisfies global space-time Strichartz estimates if the trapped set is empty or sufficiently filamentary, and we derive a scattering theory for the L^2 critical nonlinear Schrödinger equation in this geometric framework.

***Résumé* (Inégalités de Strichartz globales en temps sur des variétés asymptotiquement plates à capture tempérée)**

Nous démontrons des inégalités de Strichartz pour l'équation de Schrödinger sur une grande famille de variétés asymptotiquement coniques. Si P est l'opérateur de Laplace et $f_0 \in C_0^\infty(\mathbb{R})$ une fonction de troncature égale à 1 près de zéro, nous montrons d'abord que la partie basse fréquence de toute solution $e^{-itP}u_0$, i.e., $f_0(P)e^{-itP}u_0$, satisfait les mêmes inégalités de Strichartz que sur $\mathbb{R}^n$, en dimension $n \geq 3$. Nous montrons également que la partie haute fréquence $(1-f_0)(P)e^{-itP}u_0$ vérifie également des inégalités de Strichartz sans perte de dérivée à l'extérieur d'un compact, même si la variété possède des géodésiques captées mais dans un sens tempéré. Nous montrons ensuite que la solution complète $e^{-itP}u_0$ satisfait des inégalités de Strichartz globales en espace-temps à condition que l'ensemble capté soit vide ou suffisamment fin, et nous obtenons une théorie de la diffusion pour l'équation de Schrödinger non linéaire L^2 critique dans ce contexte géométrique.

CONTENTS

CHAPTER 1

INTRODUCTION AND MAIN RESULTS

In the past ten or fifteen years, a lot of activity has been devoted to study Strichartz inequalities on manifolds. We recall that these inequalities were stated first on $\mathbb{R}^n$ for the wave equation [**37**] and then the Schrödinger one [**21**]; for the Schrödinger equation and a pair $(p, q) \in [2, \infty] \times [2, \infty]$, they read

$$\|u\|_{L^p(\mathbb{R}, L^q)} \lesssim \|u_0\|_{L^2}, \qquad u(t) = e^{it\Delta} u_0, \qquad \text{if } \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (n, p, q) \neq (2, 2, \infty).$$

(A pair (p, q) satisfying the last two conditions is called Schrödinger admissible.) The strong interest on Strichartz inequalities is mainly related to their key role in the study of nonlinear dispersive equations (see, e.g., [**12, 38**]).

On compact manifolds these estimates may be different from those on $\mathbb{R}^n$, either due to the strong confinement leading to derivative losses for the Schödinger equation [**10**] (the L^2 norm of initial data is replaced by some Sobolev norm) or to the absence of global in time estimates (if initial data are eigenfunctions the solutions are periodic in time).

One may ask to which extent the estimates on $\mathbb{R}^n$ still hold on noncompact manifolds, at least in the class of asymptotically flat ones. For the Schrödinger equation, the only one considered from now on, this problem was considered in several articles for local in time estimates [**36, 35, 22, 7, 30**]. From the geometrical point of view, those papers consider stronger and stronger perturbations, namely from compactly supported perturbations of the flat metric on $\mathbb{R}^n$ to long range perturbations of conical metrics on manifolds. We refer to Definition 1.1 for a description of long range asymptotically conical metrics but point out here that long range perturbations are natural in that it is the only type of decay which is invariant under a change of radial coordinates (see [**5**]).

Global in time estimates for long range perturbations are considerably more delicate to obtain and have been considered in fewer papers [**39, 28, 23**] (see also [**8**] with a low frequency cutoff).

To prove global Strichartz inequalities on curved backgrounds, one has to face two difficulties. The first one, which does not happen on $\mathbb{R}^n$, is the possible occurring of trapped geodesics (geodesics not escaping to infinity, in the future or in the past).

This trapping is only sensitive at high frequencies and may affect the estimates by a loss of derivatives. However, if it is sufficiently weak, one can still expect Strichartz estimates without loss as shown in **[11]** locally in time. Trapping is already a problem for local in time estimates hence a fortiori for global in time ones.

The second difficulty stems in the analysis of low frequencies. Indeed, except in a few model situations such as $\mathbb{R}^n$ or flat cones **[20]** where the fundamental solution of the Schrödinger equation can be computed explicitly, the only robust strategy accessible so far is to localize the solution in frequency, e.g., by mean of a Littlewood-Paley decomposition, and then to prove Strichartz estimates for the spectrally localized components by using microlocal techniques to derive appropriate dispersive estimates. Due to the uncertainty principle, low frequency data cannot be studied purely by microlocal techniques and thus require additional non trivial estimates. On $\mathbb{R}^n$ (or a pure cone), one may use a global scaling argument to reduce the analysis of low frequency blocks to the study at frequency one, but this is in general impossible on manifolds.

The first breakthrough on global in time Strichartz estimates was done by Tataru in **[39]** where he considered long range and globally small perturbations of the Euclidean metric, with C^2 and time dependent coefficients. In this framework, no trapping could occur. The results were then improved in **[28]** by allowing more general perturbations in a compact set, including some weak trapping. Recently, Hassell-Zhang **[23]** partially extended those results by considering the general geometric framework of asymptotically conic manifolds and including very short range potentials, but using a non-trapping condition.

In the present paper, we improve on those references in the following directions. On one hand, we consider a class of asymptotically conic manifolds which is larger than the one of Hassell-Zhang, and contains all usual smooth long range perturbations of the Euclidean metric. More importantly, we allow the possibility to have trapped trajectories and, assuming this trapping to be temperate (assumption (1.5)), show that the solutions to the linear Schrödinger equation enjoy the same global in time estimates without loss as on $\mathbb{R}^n$ outside a large enough compact set. This fact is a priori not clear at all since, by the infinite speed of propagation of the Schrödinger equation, one may fear that the geometry and the form of the initial datum inside a compact set has an influence on the solution all the way to spatial infinity. This question was considered first in **[7]** locally in time and then in **[28]** globally in time case but our approach in this paper allows to deal with much stronger types of trapping than in this last reference (see the discussion after Theorem 1.3).

As a byproduct of this analysis, we derive global space-time Strichartz estimates without loss if there is no trapping (thus recovering the results of Hassell-Zhang for a larger class of manifolds, when there is no potential) or if the trapping is filamentary in the sense of **[33, 11]**. In particular, we extend to the global in time case one of the results of **[11]**.

Then, we apply these estimates to the scattering theory of the L^2 critical nonlinear Schrödinger equation with small data on a manifold with filamentary (or empty) trapped set (Theorem 9.1).

From the technical point of view, an important part of our paper is devoted to construct tools adapted to the analysis of low frequencies. In particular, along the way, we develop a new version of the Isozaki-Kitada parametrix for long range metrics. Recall that the Isozaki-Kitada parametrix was introduced on $\mathbb{R}^n$ to study the scattering theory of Schrödinger operators with long range potentials [**24**]. One of the new features of our parametrix is the treatment of low frequencies which, to our knowledge, does not seem to have been much considered before, up to the reference [**16**] in the context of scattering by potentials on $\mathbb{R}^n$ which is very different from ours (especially at low energy). We derive related L^2 propagation estimates which are needed in the present paper but can be of interest for other questions of scattering theory, such as the study of scattering matrices at low energy. In a more directly oriented PDE perspective, the methods developed in this paper also allow to handle other dispersive models like fractional equations [**18**].

Let us now state our results more precisely.

Let $(\mathcal{M}, G)$ be an asymptotically conic manifold, possibly with a boundary, i.e., a manifold diffeomorphic away from a compact set to a product $(R_{\mathcal{M}}, +\infty) \times \mathcal{S}$, for some closed Riemannian manifold $(\mathcal{S}, \bar{g})$, such that G is a long range perturbation of the exact conical metric $dr^2 + r^2\bar{g}$. To state a precise definition, we denote by $\Gamma(T^p_q\mathcal{S})$ the space of (p,q) tensors on $\mathcal{S}$, i.e., sections of $(\bigotimes^p T\mathcal{S}) \otimes (\bigotimes^q T^*\mathcal{S})$, and for a given smooth map $e = e(r)$ defined on $(R_{\mathcal{M}}, +\infty)$ with values in $\Gamma(T^p_q\mathcal{S})$, we will note

$$e \in S^{-\nu} \qquad \Longleftrightarrow \qquad N_{pq}\big(\partial_r^j e(r)\big) \lesssim \langle r \rangle^{-\nu-j}$$

for each semi-norm N_{pq} of $\Gamma(T^p_q\mathcal{S})$ and $j \geq 0$. If $(\theta_1, \ldots, \theta_{n-1})$ are local coordinates on $\mathcal{S}$, this means equivalently that e is a linear combination of terms of the form $e^{j_1\cdots j_p}_{i_1\cdots i_q}(r,\theta) d\theta_{i_1} \otimes \cdots \otimes d\theta_{i_q} \otimes \partial_{\theta_{j_1}} \otimes \cdots \otimes \partial_{\theta_{j_p}}$ such that, for each j and α, we have an estimate $|\partial_r^j \partial_\theta^\alpha e^{j_1\cdots j_p}_{i_1\cdots i_q}(r,\theta)| \lesssim \langle r \rangle^{-\nu-j}$ locally uniformly in θ (see also the paragraph *Standard symbol classes* in Chapter 2). Here $\langle \cdot \rangle$ is the standard Japanese bracket.

Definition 1.1. – *A Riemannian manifold $(\mathcal{M}, G)$ is* asymptotically conic *if it is connected and if there exist a continuous and proper function $r : \mathcal{M} \to [0, +\infty)$, a compact subset $\mathcal{K} \Subset \mathcal{M}$ and a closed Riemannian manifold $(\mathcal{S}, \bar{g})$ such that for some $R_{\mathcal{M}} > 0$ there is a diffeomorphism*

$$\Omega : \mathcal{M} \setminus \mathcal{K} \ni m \mapsto \big(r(m), \omega(m)\big) \in (R_{\mathcal{M}}, +\infty) \times \mathcal{S}$$

through which

$$G = \Omega^*\big(A(r)dr^2 + 2rB(r)dr + r^2 g(r)\big),$$

where $A(r) \in \Gamma(T^0_0\mathcal{S})$, $B(r) \in \Gamma(T^0_1\mathcal{S})$ and $g(r) \in \Gamma(T^0_2\mathcal{S})$ is a Riemannian metric on $\mathcal{S}$ such that, for some $\nu > 0$,

$$A - 1 \in S^{-\nu}, \qquad B \in S^{-\nu}, \qquad g(\cdot) - \bar{g} \in S^{-\nu}. \tag{1.1}$$

If $A \equiv 1$ and $B \equiv 0$, one says the metric G is in normal form.

Without loss of generality, we will assume that G is in normal form (see Appendix A). This plays no role in the present introduction but will be useful in later chapters.

Everywhere in the sequel, we denote by $L^q(\mathcal{M})$ or just L^q the Lebesgue spaces associated to the Riemannian measure on $\mathcal{M}$. We let P be the Friedrichs extension of $-\Delta_G$ on $L^2(\mathcal{M})$, namely the unique selfadjoint realization if $\mathcal{M}$ has no boundary or the Dirichlet one if $\partial\mathcal{M}$ is not empty. One interest of our geometric framework is that, if $n \geq 3$, we have a Sobolev estimate

$$\|v\|_{L^{2^*}(\mathcal{M})} \lesssim \|P^{1/2}v\|_{L^2(\mathcal{M})}, \qquad 2^* = \frac{2n}{n-2}, \tag{1.2}$$

for all v in the domain of $P^{1/2}$ (see Appendix C for a proof).

For $u_0 \in L^2(\mathcal{M})$, we let $u(t) := e^{-itP}u_0$ be the solution to the Schrödinger equation

$$i\partial_t u - Pu = 0, \qquad u|_{t=0} = u_0.$$

Let $f_0 \in C_0^\infty(\mathbb{R})$ be such that $f_0 \equiv 1$ on $[-1,1]$ and split $u(t) = u_{\text{low}}(t) + u_{\text{high}}(t)$ according to low and high frequencies, i.e.,

$$u_{\text{low}}(t) := f_0(P)e^{-itP}u_0, \qquad u_{\text{high}}(t) = (1-f_0)(P)e^{-itP}u_0. \tag{1.3}$$

Theorem 1.2 (Global space-time low frequency estimates). – *Assume that $n \geq 3$ and let (p,q) be a Schrödinger admissible pair. Then there exists $C > 0$ such that, for all $u_0 \in L^2(\mathcal{M})$,*

$$\|u_{\text{low}}\|_{L^p(\mathbb{R};L^q(\mathcal{M}))} \leq C\|u_0\|_{L^2(\mathcal{M})}. \tag{1.4}$$

Notice that in this theorem $\partial\mathcal{M}$ may be empty or not.

Proof. – Section 8.2. □

Theorem 1.3 (Global in time high frequency estimates at spatial infinity). – *Assume that $n \geq 2$ and that for some $M > 0$ large enough, we have for all $\chi \in C_c^\infty(\mathcal{M})$*

$$\|\chi(P-\lambda \pm i0)^{-1}\chi\|_{L^2(\mathcal{M})\to L^2(\mathcal{M})} \lesssim_\chi \lambda^M, \qquad \lambda \geq 1. \tag{1.5}$$

Then there exists $R \gg 1$ such that for any Schrödinger admissible pair (p,q) there exists $C > 0$ such that

$$\|1_{\{r>R\}}u_{\text{high}}\|_{L^p(\mathbb{R};L^q(\mathcal{M}))} \leq C\|u_0\|_{L^2(\mathcal{M})}, \tag{1.6}$$

for all $u_0 \in L^2(\mathcal{M})$.

If we recast the global in time estimates at spatial infinity of [**28**, Theorem 1.5] in our framework, these authors show that

$$\|1_{\{r>R\}}u_{\text{high}}\|_{L^p(\mathbb{R};L^q)} \lesssim \|u_0\|_{L^2} + \|1_{\{r<R\}}u_{\text{high}}\|_{L^2(\mathbb{R};L^2)},$$

where the last term can be controlled by $\|u_0\|_{L^2}$ thanks to (1.5) if $M \leq 0$ (the usual non-trapping case is $M = -1/2$) but not clearly otherwise. In our result, the right hand side of (1.6) does not involve any corrective term depending on u and holds for any M.

Note that examples of situations where bounds of the form (1.5) hold include [**33, 14**] in some trapping geometries and, of course, the non-trapping case [**42**].

We also remark that, as in Theorem 1.2, the boundary of $\mathcal{M}$ does not need to be empty but this observation is less relevant here for we consider estimates near infinity.

Theorems 1.2 and 1.3 reduce the proof of Strichartz estimates on u to estimates on $1_{\{r\leq R\}}u_{\rm high}$. The interest is that it is relatively easy to plug some results or techniques proved locally in space to derive global estimates. Here we consider the classical example of a non trapping manifold, but also include the case of weakly trapping geodesic flow.

Let $\mathcal{T} \subset T^*\mathcal{M}$ be the trapped set of the geodesic flow and $\pi(\mathcal{T}) \Subset \mathcal{M}$ be its projection onto the base space. We need the following condition on $\mathcal{T}$.

Assumption 1.4 (Weak trapping condition). – *We assume the following conditions introduced in* [**11**]*:*

- *the manifold $(\mathcal{M}, G)$ is a scattering manifold* [**29, 15**]*,*
- *there exists an open set $\mathcal{M}_- \subset \mathcal{M}$ containing $\pi(\mathcal{T})$ which can be extended to a complete manifold with sectional curvatures bounded above by a negative constant,*
- *$\mathcal{M}_-$ is geodesically convex in the sense that any geodesic entering $\pi^{-1}(\mathcal{M}\backslash\mathcal{M}_-)$ remains in this region thereafter,*
- *the topological pressure $P(s)$ of the trapped set $\mathcal{T}$ satisfies $P(1/2) < 0$.*

We refer to [**33**, Section 3.3] for details on the topological pressure $P(s)$.

Theorem 1.5 (Global spacetime estimates without loss). – *Assume that $n \geq 3$ and $\partial\mathcal{M}$ is empty. If either*

- *the geodesic flow is non-trapping and (p, q) is any Schrödinger admissible pair,*
- *assumption 1.4 is satisfied and (p, q) is any non endpoint Schrödinger admissible pair*

then there exists $C > 0$ such that

$$\|u\|_{L^p(\mathbb{R};L^q(\mathcal{M}))} \leq C\|u_0\|_{L^2(\mathcal{M})}, \tag{1.7}$$

for all $u_0 \in L^2(\mathcal{M})$.

This theorem improves on the result of [**23**] in two directions: Hassell-Zhang only consider the non-trapping case and, even in the non-trapping situation, we consider more general types of ends. It also provides a global in time version of the estimates of [**11**].

We state this result in the boundaryless case in order to give complete proofs or references. We emphasize however that using the techniques of [**25**] it can certainly be extended to the case when $\mathcal{M}$ has a strictly geodesically concave boundary and is non-trapping for the associated billiard flow

We recall finally the well known fact that inhomogeneous Strichartz estimates, for non endpoint pairs, can be derived from the homogeneous ones (1.7) by using the Christ-Kiselev Lemma [**13**]; this is sufficient for the applications to the nonlinear equations studied in Section 9.

Here is the plan of our paper. In Chapter 2, we record notation about charts, partitions of unity, scaling operators, etc. that will be used in further chapters. In Chapter 3, we describe the pseudo-differential calculus adapted to our framework, including a rescaled one for low frequency estimates which is not quite standard. In Chapter 4, we prove Littlewood-Paley decompositions at low and high frequencies. In Chapters 5 and 6, we construct an Isozaki-Kitada parametrix for the microlocalized Schrödinger group, both at high and low frequencies. We use it in Chapter 7 to derive some L^2 propagation estimates to be used in Chapter 8 where the theorems stated in this introduction are proved. Finally, in Chapter 9, we give nonlinear applications of our Strichartz estimates.

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