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0 Introduction

Tensor products of C*-algebras have been extensively studied over the last decades (see references in [12]). One of the main results was obtained by M. Takesaki in [15] where he proved that the spatial tensor product $A \otimes_{\min} B$ of two C*-algebras A and B always defines the minimal C*-norm on the algebraic tensor product $A \otimes_{alg} B$ of A and B over the complex field $\mathbf{C}$.

More recently, G.G. Kasparov constructed in [10] a tensor product over $C(X)$ for $C(X)$-algebras. The author was also led to introduce in [3] several notions of tensor products over $C(X)$ for $C(X)$-algebras and to study the links between those objects.

Notice that E. Kirchberg and S. Wassermann have proved in [11] that the subcategory of continuous fields over a Hausdorff compact space is not closed under such tensor products over $C(X)$ and therefore, in order to study tensor products over $C(X)$ of continuous fields, it is natural to work in the $C(X)$-algebras framework.

Let us introduce the following definition:

DEFINITION 0.1 *Given two $C(X)$-algebras A and B, we denote by $\mathcal{I}(A, B)$ the involutive ideal of the algebraic tensor product $A \otimes_{alg} B$ generated by the elements $(fa) \otimes b - a \otimes (fb)$, where $f \in C(X)$, $a \in A$ and $b \in B$.*

Our aim in the present article is to study the C*-norms on the algebraic tensor product $(A \otimes_{alg} B)/\mathcal{I}(A, B)$ of two $C(X)$-algebras A and B over $C(X)$ and to see how one can enlarge the results of Takesaki to this framework.

We first define an ideal $\mathcal{J}(A, B) \subset A \otimes_{alg} B$ which contains $\mathcal{I}(A, B)$ such that every C*-semi-norm on $A \otimes_{alg} B$ which is zero on $\mathcal{I}(A, B)$ is also zero on $\mathcal{J}(A, B)$ and we prove that there always exist a minimal C*-norm $\| \, \|_m$ and a maximal C*-norm $\| \, \|_M$ on the quotient $(A \otimes_{alg} B)/\mathcal{J}(A, B)$.

We then study the following question of G.A. Elliott ([5]): when do the two ideals $\mathcal{I}(A, B)$ and $\mathcal{J}(A, B)$ coincide?

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1 Preliminaries

We briefly recall here the basic properties of $C(X)$-algebras.

Let X be a Hausdorff compact space and $C(X)$ be the C*-algebra of continuous functions on X. For $x \in X$, define the morphism $e_x : C(X) \to \mathbf{C}$ of evaluation at x and denote by $C_x(X)$ the kernel of this map.

DEFINITION 1.1 *([10]) A $C(X)$-algebra is a* C*-*algebra A endowed with a unital morphism from $C(X)$ in the center of the multiplier algebra $M(A)$ of A.*

We associate to such an algebra the unital $C(X)$-algebra $\mathcal{A}$ generated by A and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$.

For $x \in X$, denote by A_x the quotient of A by the closed ideal $C_x(X)A$ and by a_x the image of $a \in A$ in the fibre A_x. Then, as

$$\|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\},$$

the map $x \mapsto \|a_x\|$ is upper semi-continuous for all $a \in A$ ([14]).

Note that the map $A \to \oplus A_x$ is a monomorphism since if $a \in A$, there is a pure state ϕ on A such that $\phi(a^*a) = \|a\|^2$. As the restriction of ϕ to $C(X) \subset M(A)$ is a character, there exists $x \in X$ such that ϕ factors through A_x and so $\phi(a^*a) = \|a_x\|^2$.

Let $S(A)$ be the set of states on A endowed with the weak topology and let $\mathcal{S}_X(A)$ be the subset of states φ whose restriction to $C(X) \subset M(A)$ is a character, i-e such that there exists an $x \in X$ (denoted $x = p(\varphi)$) verifying $\varphi(f) = f(x)$ for all $f \in C(X)$. Then the previous paragraph implies that the set of pure states $P(A)$ on A is included in $\mathcal{S}_X(A)$.

Let us introduce the following notation: if $\mathcal{E}$ is a Hilbert A-module where A is a C*-algebra, we will denote by $\mathcal{L}_A(\mathcal{E})$ or simply $\mathcal{L}(\mathcal{E})$ the set of bounded A-linear operators on $\mathcal{E}$ which admit an adjoint ([9]).

DEFINITION 1.2 *([3]) Let A be a $C(X)$-algebra.*

A $C(X)$-representation of A in the Hilbert $C(X)$-module $\mathcal{E}$ is a morphism $\pi : A \to \mathcal{L}(\mathcal{E})$ which is $C(X)$-linear, i.e. such that for every $x \in X$, the representation $\pi_x = \pi \otimes e_x$ in the Hilbert space $\mathcal{E}_x = \mathcal{E} \otimes_{e_x} \mathbf{C}$ factors through a representation of A_x. Furthermore, if π_x is a faithful representation of A_x for every $x \in X$, π is said to be a field of faithful representations of A.

A continuous field of states on A is a $C(X)$-linear map $\varphi : A \to C(X)$ such that for any $x \in X$, the map $\varphi_x = e_x \circ \varphi$ defines a state on A_x.

If π is a $C(X)$-representation of the $C(X)$-algebra A, the map $x \mapsto \|\pi_x(a)\|$ is lower semi-continuous since $\langle \xi, \pi(a)\eta \rangle \in C(X)$ for every $\xi, \eta \in \mathcal{E}$. Therefore, if A admits a field of faithful representations π, the map $x \mapsto \|a_x\| = \|\pi_x(a)\|$ is continuous for every $a \in A$, which means that A is a continuous field of C*-algebras over X ([4]).

The converse is also true ([3] théorème 3.3): given a separable $C(X)$-algebra A, the following assertions are equivalent:

1. A is a continuous field of C*-algebras over X,

2. the map $p : \mathcal{S}_X(A) \to X$ is open,

3. A admits a field of faithful representations.

2 C*-norms on $(A \otimes_{alg} B)/\mathcal{J}(A,B)$

DEFINITION 2.1 *Given two $C(X)$-algebras A and B, we define the involutive ideal $\mathcal{J}(A,B)$ of the algebraic tensor product $A \otimes_{alg} B$ of elements $\alpha \in A \otimes_{alg} B$ such that $\alpha_x = 0$ in $A_x \otimes_{alg} B_x$ for every $x \in X$.*

By construction, the ideal $\mathcal{I}(A,B)$ is included in $\mathcal{J}(A,B)$.

PROPOSITION 2.2 *Assume that $\|\ \|_\beta$ is a C*-semi-norm on the algebraic tensor product $A \otimes_{alg} B$ of two $C(X)$-algebras A and B.*

If $\|\ \|_\beta$ is zero on the ideal $\mathcal{I}(A,B)$, then

$$\|\alpha\|_\beta = 0 \text{ for all } \alpha \in \mathcal{J}(A,B).$$

Proof: Let D_β be the Hausdorff completion of $A \otimes_{alg} B$ for $\|\ \|_\beta$. By construction, D_β is a quotient of $A \otimes_{\max} B$. Furthermore, if C_Δ is the ideal of $C(X \times X)$ of functions which are zero on the diagonal, the image of C_Δ in $M(D_\beta)$ is zero.

As a consequence, the map from $A \otimes_{\max} B$ onto D_β factors through the quotient $A \overset{M}{\otimes}_{C(X)} B$ of $A \otimes_{\max} B$ by $C_\Delta \times (A \otimes_{\max} B)$.

But an easy diagram-chasing argument shows that $(A \overset{M}{\otimes}_{C(X)} B)_x = A_x \otimes_{\max} B_x$ for every $x \in X$ ([3] corollaire 3.17) and therefore the image of $\mathcal{J}(A,B) \subset A \otimes_{\max} B$ in $A \overset{M}{\otimes}_{C(X)} B$ is zero. □

2.1 The maximal C*-norm

DEFINITION 2.3 *Given two $C(X)$-algebras A_1 and A_2, we denote by $\|\ \|_M$ the C*-semi-norm on $A_1 \otimes_{alg} A_2$ defined for $\alpha \in A_1 \otimes_{alg} A_2$ by*

$$\|\alpha\|_M = \sup\{\|(\sigma_1^x \otimes_{\max} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \to (A_i)_x$.

As $\|\ \|_M$ is zero on the ideal $\mathcal{J}(A_1, A_2)$, if we identify $\|\ \|_M$ with the C*-semi-norm induced on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$, we get:

PROPOSITION 2.4 *The semi-norm $\|\ \|_M$ is the maximal C*-norm on the quotient $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: By construction, $\|\ \|_M$ defines a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Moreover, as the quotient $A_1 \overset{M}{\otimes}_{C(X)} A_2$ of $A_1 \otimes_{\max} A_2$ by $C_\Delta \times (A_1 \otimes_{\max} A_2)$ maps injectively in

$$\underset{x \in X}{\oplus} (A_1 \overset{M}{\otimes}_{C(X)} A_2)_x = \underset{x \in X}{\oplus} ((A_1)_x \otimes_{\max} (A_2)_x),$$

the norm of $A_1\overset{M}{\otimes}_{C(X)}A_2$ coincides on the dense subalgebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ with $\| \|_M$. But we saw in proposition 2.2 that if $\| \|_\beta$ is a C*-norm on the algebra $(A \otimes_{alg} B)/\mathcal{J}(A, B)$, the completion of $(A \otimes_{alg} B)/\mathcal{J}(A, B)$ for $\| \|_\beta$ is a quotient of $A\overset{M}{\otimes}_{C(X)}B$. □

2.2 The minimal C*-norm

DEFINITION 2.5 *Given two $C(X)$-algebras A_1 and A_2, we define the semi-norm $\| \|_m$ on $A_1 \otimes_{alg} A_2$ by the formula*

$$\|\alpha\|_m = \sup\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \to (A_i)_x$ and we denote by $A_1\overset{m}{\otimes}_{C(X)}A_2$ the Hausdorff completion of $A_1 \otimes_{alg} A_2$ for that semi-norm.

Remark: In general, the canonical map $(A_1\overset{m}{\otimes}_{C(X)}A_2)_x \to (A_1)_x \otimes_{\min} (A_2)_x$ is not a monomorphism ([11]).

By construction, $\| \|_m$ induces a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. We are going to prove that this C*-norm defines the minimal C*-norm on the involutive algebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Let us introduce some notation.

Given two unital $C(X)$-algebras A_1 and A_2, let $P(A_i) \subset \mathcal{S}_X(A_i)$ denote the set of pure states on A_i and let $P(A_1) \times_X P(A_2)$ denote the closed subset of $P(A_1) \times P(A_2)$ of couples (ω_1, ω_2) such that $p(\omega_1) = p(\omega_2)$, where $p : P(A_i) \to X$ is the restriction to $P(A_i)$ of the map $p : \mathcal{S}_X(A_i) \to X$ defined in section 1.

LEMMA 2.6 *Assume that $\| \|_\beta$ is a C*-semi-norm on the algebraic tensor product $A_1 \otimes_{alg} A_2$ of two unital $C(X)$-algebras A_1 and A_2 which is zero on the ideal $\mathcal{J}(A_1, A_2)$ and define the closed subset $S_\beta \subset P(A_1) \times_X P(A_2)$ of couples (ω_1, ω_2) such that*

$$\Big|(\omega_1 \otimes \omega_2)(\alpha)\Big| \leq \|\alpha\|_\beta \text{ for all } \alpha \in A_1 \otimes_{alg} A_2.$$

If $S_\beta \neq P(A_1) \times_X P(A_2)$, there exist self-adjoint elements $a_i \in A_i$ such that $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Proof: Define for $i = 1, 2$ the adjoint action ad of the unitary group $\mathcal{U}(A_i)$ of A_i on the pure states space $P(A_i)$ by the formula

$$[(ad_u)\omega](a) = \omega(u^*au).$$

Then S_β is invariant under the product action $ad \times ad$ of $\mathcal{U}(A_1) \times \mathcal{U}(A_2)$ and we can therefore find non empty open subsets $U_i \subset P(A_i)$ which are invariant under the action of $\mathcal{U}(A_i)$ such that $(U_1 \times U_2) \cap S_\beta = \emptyset$.

Now, if K_i is the complement of U_i in $P(A_i)$, the set

$$K_i^\perp = \{a \in A_i \mid \omega(a) = 0 \text{ for all } \omega \in K_i\}$$