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ALMOST ÉTALE EXTENSIONS

by

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Abstract. — The theory of almost étale coverings allows to compare crystalline and p-adic étale cohomology, for schemes over a p-adic discrete valuation ring. Using Frobenius the main technical result (a purity theorem) is reproved and extended to all toroidal singularities. As a consequence one obtains Tsuji's comparison theorem for schemes with such type of singularities, even for cohomology with coefficients in suitable local systems. On the way we have to establish some basic results on finiteness of crystalline cohomology with such coefficients.

0. Introduction

The theory of almost étale coverings goes back to Tate for the case of discrete valuation rings. The higher dimensional theory has been invented by the author, and used to prove conjectures about Hodge-Tate and crystalline structures on the étale cohomology of varieties over p-adic fields. The main purpose of these notes is to extend the theory to cover toroidal singularities, especially all schemes with semistable reduction. This is possible because we have found a new method of proof for the main technical result, the purity theorem. This new method makes heavy use of Frobenius and of toroidal geometry. After starting and proving the new purity theorem we study duality, first local and then global. For the globalization we need a topos $\mathfrak{X}$ which (X a scheme or an algebraic space over a p-adic discrete valuation ring V, X_K its generic fibre) is an étale localization (in X) of the topos of locally constant sheaves on X_K. Thus its cohomology (or better the direct images to the étale topos of X) is Galois-cohomology. Our previous local theory applies to "coherent sheaves" on this topos, and they satisfy finiteness, Künneth and Poincaré-duality like any decent theory. However for this we have to work in the almost category, that is we divide by the subcategory of sheaves annihilated by any positive (fractional) power of p. A

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more systematic study of almost mathematics can be found in the recent preprint [GR] by O. Gabber and L. Ramero.

It is quite remarkable that one can still apply the Artin-Schreier exact sequence to derive from the results for "discrete" coefficients, like $\mathbb{Z}/p^s\mathbb{Z}$. Namely for these coefficients one cannot use the almost category. However at one key turn almost invariants under Frobenius coincides with real invariants. Let us also mention that we make again heavy use of Frobenius (now on the coefficients) and Fontaine's construction of rings $A_{\mathrm{inf}}(R)$. Finally we show that the theory with discrete coefficients gives étale cohomology of the generic fibre X_K. For this we have to compare characteristic classes or better Gysin maps. For regular embeddings one reduces to codimension one and an explicit calculation of local direct images. The latter replaces a more global (and less transparent) argument in [Fa 3]. In general one uses toroidal modifications to make embeddings regular. That is possible for the diagonal was a great surprise for the author.

At the end we explain how to derive comparison maps to crystalline cohomology. However this is only sketched as all these ideas have already been presented in a number of papers ([Fa 3], [Fa 5], [Fa 6], [Fa 7], so that everybody interested in the matter can learn it there. We also add some (a little bit) incomplete results about cohomology of isocrystals, which we could not find in the literature.

All in all this suffices to prove the C_{st}-conjecture of Fontaine-Jansen. This had been done previously by T. Tsuji ([Ts]). However we can also treat the case of non-constant coefficients where some new phenomena occur. Anyway the author profited from his participation in the p-adic year at Institut Poincaré in Paris, first by giving lectures on a preliminary version and then by learning about T. Tsuji's technique. I have to thank the referee who found many mistakes and made an enormous number of suggestions for improvements.

1. Almost-Mathematics

Let $\overline{V}$ denote a (commutative with unit) ring together with a sequence of principal ideals $\mathfrak{m}_\alpha \subseteq \overline{V}$ parametrized by positive elements $\alpha \in \Lambda^+$, where $\Lambda \subseteq \mathbb{Q}$ denotes a subgroup dense in $\mathbb{R}$. Denote by π a generator of $\mathfrak{m}_1$, and π^α a generator of $\mathfrak{m}_\alpha$. Assume furthermore that $\pi^\alpha \pi^\beta = \text{unit} \cdot \pi^{\alpha+\beta}$, and that π^α is not a zero-divisor.

Examples

a) $\overline{V} = \overline{\mathbb{Z}}_p =$ integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$, $\mathfrak{m}_\alpha =$ elements of valuation $\geqslant \alpha$, $\pi = p$ ($\Lambda = \mathbb{Q}$).

b) Consider $R = \varprojlim(\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p, \text{Frobenius})$, $\overline{V} = W(R)$,

$$\pi = [\underline{p}, 0, \dots, 0] \in W(R),$$

$\underline{p} = \varprojlim p^{1/p^n} \in R$, $\Lambda = \mathbb{Z}[1/p]$, $\pi^\alpha =$ the obvious element.

If $\mathfrak{m} = \bigcup_{\alpha>0} \mathfrak{m}_\alpha$, then a $\overline{V}$-module is almost zero if it is annihilated by $\mathfrak{m}$. Denote this by $M \approx 0$. Assume R is a $\overline{V}$-algebra and M an R-module.

1. Definition. — M is almost projective if $\mathrm{Ext}^i_R(M,N) \approx 0$, all R-modules N and all $i > 0$ (or only $i = 1$)
M is almost flat if $\mathrm{Tor}_i^R(M,N) \approx 0$, all R-modules N and all $i > 0$ (or only $i = 1$)
M is almost finitely generated if for each $\alpha > 0, \alpha \in \Lambda$, there exists a finitely generated R-module N and a π^α-isomorphism $\psi_\alpha : N \to M$ (i.e., there exists $\phi_\alpha : M \to N$ with $\phi_\alpha \circ \psi_\alpha = \pi^\alpha \cdot \mathrm{id}_N$, $\psi_\alpha \circ \phi_\alpha = \pi^{\alpha_0} \circ \mathrm{id}_M$).
M is almost finitely presented: dito with N finitely presented.

2. Remarks

a) P almost projective
$\Leftrightarrow$ for all surjections $f : M \to N$ and maps $g : P \to N$, $\pi^\alpha \cdot g$ factors through f $(\alpha > 0,\ \alpha \in \Lambda)$
$\Leftrightarrow$ all $\alpha > 0$, $\alpha \in \Lambda$, there exists a free R-module L and maps $L \xrightarrow{g} P \xrightarrow{f} L$ with $g \circ f = \pi^\alpha \cdot \mathrm{id}_p$.

b) If P is almost projective and almost finitely generated, then P is almost finitely presented. Then also $P^* = \mathrm{Hom}_R(P,R), P \otimes_R P, \Lambda^i P$ etc. are almost finitely generated projective. Furthermore there is an almost isomorphism $P \otimes_R P^* \approx \mathrm{End}_R(P)$, and thus a trace-map $\mathrm{tr} : \mathfrak{m} \otimes_R \mathrm{End}_R(P) \to R$, or equivalently $\mathrm{End}_R(P) \to \mathrm{Hom}(\mathfrak{m}, R)$. Thus for $f \in \mathrm{End}_R(P)$ we can define a power-series

$$\det(1 + Tf) = \sum_{i=0}^{\infty} \mathrm{tr}(\Lambda^i f) \cdot T^i \in \mathrm{Hom}(\mathfrak{m}, R)[[T]].$$

One checks that $\mathrm{Hom}(\mathfrak{m}, R)$ is a ring

$$f \circ g(\pi^{\alpha+\beta}) = f(\pi^\alpha) \cdot g(\pi^\beta))$$

and that

$$\det(1 + Tf)\det(1 + Tg) = \det(1 + T(f + g + Tfg))(= \sum_{i=0}^{\infty} \mathrm{tr}(\Lambda^i(f + g + Tfg)) \cdot T^i$$

We say that P has rank $\leqslant r$ if $\Lambda^{r+1} P \approx 0$. Then letting $e \in \mathrm{Hom}(\mathfrak{m}, R)$ denote the coefficient of T^r in $\det(1 + T \cdot \mathrm{id}_p)$, one checks that $e = e^2$ is an idempotent, and thus $R \approx R_1 \times R_2$ factors (up to almost isomorphism). Also $P \approx (P \otimes_R R_1) \times (P \otimes_R R_2)$.

Over one of the factors (say R_1) $e = 1$. One then checks that after base-change to R_1, $L = \Lambda^r P$ satisfies $L \otimes_R L^* \approx R$, i.e. L behaves like a line-bundle: There are the maps

$$\mathrm{tr} : \mathrm{End}_R(L) \approx L \otimes_R L^* \longrightarrow R$$

and the identity: $R \to \mathrm{End}_R(L)$. We have $\mathrm{tr}(\mathrm{id}_p) = e = 1$. In fact for two $f, g \in \mathrm{End}_R(P)$ we have

$$\mathrm{tr}(\Lambda^r f) \cdot \mathrm{tr}(\Lambda^r g) = \mathrm{tr}(\Lambda^r(fg)) = \text{coefficient of } T^{2r} \text{ in } \det((1+Tf)(1+Tg)).$$

Apply this to f and g of the form

$$f(x) = \sum_{i=1}^{r} \lambda_i(x)\mu_i, \quad \lambda_i \in P^*,\ \mu_i \in P$$
$$g(x) = \sum_{i=1}^{r} \nu_i(x)\phi_i$$

to get $\mathrm{tr}(\Lambda^r f) \cdot \det(\nu_i(\phi_j)) = \det(\nu_i(f(\phi_j)))$, i.e.

$$\mathrm{tr}(\lambda^r f)(\nu_1 \wedge \cdots \wedge \nu_r \mid \phi_1 \wedge \cdots \wedge \phi_r) = (\nu_1 \wedge \cdots \wedge \nu_r \mid \Lambda^r f(\phi_1 \wedge \cdots \wedge \phi_r)).$$

Varying ν's and ϕ's gives $\Lambda^r f = \mathrm{tr}(\Lambda^r f)$ is a scalar. As the $\Lambda^r f$ (given by $x \mapsto \langle \lambda_1 \wedge \cdots \wedge \lambda_r \mid x(\mu_1 \wedge \cdots \wedge \mu_r)\rangle$ generate $\mathrm{End}_R(L)$, everything follows. The same argument gives:
If $e = 0$, then $\mathrm{tr}(\Lambda^r f) = 0$ and $\Lambda^r f = 0$, and then $\Lambda^r P = 0$. Thus by induction we get an almost isomorphism

$$R \approx R_0 \times \cdots \times R_r,$$

such that over R_i $\Lambda^i P$ is invertible, $\Lambda^{i+1}P = 0$. Also over R_i we have a determinant-function

$$\det(f) = \mathrm{tr}(\Lambda^i f) : \mathrm{End}_R(P) \longrightarrow \mathrm{Hom}(\mathfrak{m}, R_i)$$

homogeneous of degree i with $\det(fg) = \det(f)\det(g), \det(1) = 1$.

3. Remark. — I do not know whether for arbitrary finitely generated projective P's we have a decomposition $R \approx \prod_{i=0}^{\infty} R_i$ with $P \otimes_R R_i$ of rank i.

2. Almost étale coverings

1. Definition. — A ring homomorphism $A \to B$ is called an almost étale covering if
i) B is almost finitely generated projective as an A-module, of finite rank
ii) B is almost finitely generated projective as a $B \otimes_A B$-module

2. Remarks

a) There exists an idempotent $e_{B/A} \in \mathrm{Hom}(\mathfrak{m}, B \otimes_A B)$ measuring where B has rank 1 as $B \otimes_A B$-module. $e_{B/A}$ is annihilated by $(b \otimes_A 1 - 1 \otimes_A b)$ and maps to $1 \in \mathrm{Hom}(\mathfrak{m}, B)$ under multiplication.

b) Replacing A by $\mathrm{Hom}(\mathfrak{m}, A)$ we may reduce (in many cases) to the case where $\mathrm{rank}_A B = r$ is constant. Then there exists an almost faithfully flat base-change $A \to A'$ (A' is almost flat, and $M \otimes_A A' \approx 0$ implies $M \approx 0$) with $B' = B \otimes_A A' \approx A'^r$.