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RANDOM ZONAL EIGENFUNCTIONS AND A HÖLDER VERSION OF THE PALEY–ZYGMUND THEOREM ON COMPACT MANIFOLDS

by Pierre Brun, Rafik Imekraz & Guillaume Poly

Abstract. — We study the convergence of Gaussian random series of radial/zonal eigenfunctions of the Laplace–Beltrami operator (on the Euclidean space and on the round sphere). More precisely, we obtain a simple, necessary and sufficient condition of almost sure uniform convergence (we thus complete an analysis of Ayache and Tzvetkov). In dimension 2, our strategy turns out to be linked with Hölder regularities. As a by-product, we also prove a Hölder version of the Paley–Zygmund theorem on a boundaryless Riemannian compact manifold.

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Résumé (*Fonctions propres zonales aléatoires et version höldérienne du théorème de Paley–Zygmund sur les variétés compactes*). — Nous étudions la convergence de séries aléatoires gaussiennes de fonctions propres radiales/zonales de l'opérateur de Laplace-Beltrami (sur l'espace euclidien ou sur la sphère usuelle). Plus précisément, nous obtenons une condition nécessaire et suffisante simple de convergence uniforme (presque sûre), ce qui nous permet de compléter une analyse de Ayache et Tzvetkov. En dimension deux, notre stratégie s'avère liée à la régularité höldérienne. Par conséquent, nous prouvons une version höldérienne d'un théorème de Paley-Zygmund sur une variété riemannienne compacte sans bord.

1. Introduction

Let $\mathcal{M}$ be a Riemannian manifold (we shall state more precise assumptions in the sequel) and let us consider a sequence $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(\mathcal{M})$ made of eigenfunctions of a fixed operator (for instance, a Laplace–Beltrami operator). The present paper makes a contribution in the research area where the following question is of interest: once a Banach space B of functions on $\mathcal{M}$ is fixed, can we give a necessary and sufficient condition on a sequence of coefficients $(c_k)_{k\in\mathbb{N}}$ such that the Gaussian random series $\sum g_k(\omega)c_k\phi_k$ almost surely converges in B (here, $(g_k)_{k\in\mathbb{N}}$ is a standard i.i.d. Gaussian $\mathcal{N}(0,1)$ sequence of random variables defined on an abstract probability space Ω)? Actually, such a question is too general, and we shall give a satisfactory solution in two particular cases, which we shall motivate in the present Introduction.

We prefer to refer to the Introduction of [32, 24] for more details about the history of such questions for $\mathcal{M}$ being the torus or a compact group, or the Introduction of [20] for $\mathcal{M}$ being a manifold. Here, however, we shortly recall the main contributions. One could separate the results into two directions.

Analysis on groups. — The story began with the paper of Paley and Zygmund [36], who gave a solution for random linear combinations[1] $\sum\limits_{k\in\mathbb{N}} \pm c_k e^{ikx}$. Actually, for any fixed $p\in[2,+\infty)$, such a random series belongs, with probability 1, to $L^p(-\pi,\pi)$ if and only if (c_k) belongs to $\ell^2(\mathbb{N})$. It is now known that considering random signs $\pm$ (namely i.i.d. Rademacher random variables) instead of Gaussian random variables leads to the same results[2]. The case $p=+\infty$ is much more delicate. One of the best "simple sufficient conditions" is given[3] by a result of Salem and Zygmund [37, page 291]

$$\sum_{k\geq 2}\frac{1}{k\sqrt{\ln(k)}}\sqrt{\sum_{n\geq k}|c_n|^2}<+\infty$$

1. Here, we prefer to choose an index set equaling $\mathbb{N}$ instead of the natural choice $\mathbb{Z}$ for consistency with the sequel of the article.

2. This equivalence has been proved by Marcus and Pisier.

3. Actually, such a condition may become necessary under some assumptions (see [31]).

that actually sharpens the Paley–Zygmund condition; there is $\gamma > 1$ satisfying

$$\sum_{k\geq 2} |c_k|^2 \ln^\gamma(k) < +\infty.$$

In order to obtain a condition that is sufficient and necessary on the sequence of (c_k) in the case $p = +\infty$, one needs to combine two arguments: the theorem of Dudley [10] that proves that an abstract condition, called the entropy condition, is sufficient, and then the theorem of Fernique that proves that the entropy condition is necessary (by exploiting the stationarity of Gaussian processes; see [11, pages 89–94]). Finally, the book [32] contains the complete solution for $\mathcal{M}$ being a compact group for adequate random series. We also refer to [28, Chapter 13] or [29, Chapters 3 and 6] for much more detail on this aspect.

Analysis for elliptic operators. — The story continues with the papers of Ayache–Tzvetkov [3] and Tzvetkov [38] by replacing the trigonometric functions e^{ikx} with eigenfunctions of a Laplace operator. Let us explain the contribution of Ayache and Tzvetkov with a few details; let us denote by $(Z_n^{d,\mathrm{Dir}})_{n\geq 1}$ the sequence of the radial eigenfunctions of the Laplace operator on the closed unit ball $B_d(0,1)$ of $\mathbb{R}^d$ with Dirichlet conditions for $d \geq 2$ (see (1) for the exact definition). Ayache and Tzvetkov proved that, in contrast to the trigonometric case, for any sequence of coefficients $(c_n)_{n\in\mathbb{N}^\star}$, there is an exponent[4] $p_c \in [2, +\infty]$ (depending on (c_n)) such that

$$p < p_c \Rightarrow \sum_{n\geq 1} g_n(\omega) c_n Z_n^{d,\mathrm{Dir}} \quad \text{almost surely converges in } L^p(B_d(0,1)),$$

$$p > p_c \Rightarrow \sum_{n\geq 1} g_n(\omega) c_n Z_n^{d,\mathrm{Dir}} \quad \text{almost surely diverges in } L^p(B_d(0,1)).$$

In [3, Theorem 4], it is proved that $p_c = \frac{2d}{d-2}$ in the specific case $c_n \simeq \frac{1}{n}$. Then, in [14] Grivaux obtained a simple formula giving p_c for any arbitrary sequence of coefficients $(c_n) \in \ell^2(\mathbb{N}^\star)$. Actually, the issue behind such considerations is to decide whether or not the Gaussian random series $\sum g_n(\omega) c_n Z_n^{d,\mathrm{Dir}}$ almost surely converges in $L^p(B_d(0,1))$. For any finite exponent $p > \frac{2d}{d-1}$, a complete solution is given in [18] for zonal eigenfunctions[5] on the sphere $\mathbb{S}^d$ for $d \geq 2$. In [18], the finiteness of p is very important because of interpolation arguments.

4. Actually, the inequality $p_c \geq \frac{2d}{d-1}$ holds because the eigenfunctions $Z_n^{d,\mathrm{Dir}}$ are uniformly bounded in $L^p(B_d(0,1))$ for $p < \frac{2d}{d-1}$ (we refer to [3] for explanations or the inequality in [18, page 266, line (2)]).

5. Actually, it is known that the zonal eigenfunctions on $\mathbb{S}^d$ have, in some sense, a similar behavior than that of the radial eigenfunctions $Z_n^{d,\mathrm{Dir}}$, which play a sort of canonical model of eigenfunctions that concentrate around a point (see [3, page 4428, remark d)], and this point of view will also be clarified in Section 2.

We note that a common ingredient in all the works [3, 14, 18] is the concentration of the radial (or zonal) eigenfunctions. We also refer to [22] for the analysis of radial eigenfunctions of the harmonic oscillator $-\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$.

First contribution of the paper. — The first contribution of the present paper is to find a necessary and sufficient condition for $p = +\infty$ in the zonal/radial framework studied by Ayache and Tzvetkov in [3]. Let us briefly recall the two examples of functions we have in mind.

- The sequence of radial eigenfunctions of the Laplace operator $-\Delta_{\mathrm{Dir}}$ on the Euclidean closed ball $B_d(0,1) = \{x \in \mathbb{R}^d, |x| \leq 1\}$ (for $d \geq 2$) with Dirichlet conditions. Then the sequence of radial eigenfunctions $Z_n^{d,\mathrm{Dir}}$ is given by

$$Z_n^{d,\mathrm{Dir}}(x) = c_{d,n} \frac{J_{\frac{d}{2}-1}(\lambda_{d,n}|x|)}{|x|^{\frac{d}{2}-1}}, \quad \forall x \in \mathbb{R}^d, \quad \forall n \in \mathbb{N}^\star, \tag{1}$$

  in which $\lambda_{d,n}$ is the n-th zero the Bessel function $J_{\frac{d}{2}-1}$. We have $\lambda_{d,n} \simeq n$ and $c_{d,n} \simeq \sqrt{n}$ for $n \to +\infty$ (see [3, page 4431]). The functions $Z_n^{d,\mathrm{Dir}}$ concentrate around the origin (see (22)) and satisfy the eigenfunction equation

$$-\Delta Z_n^{d,\mathrm{Dir}} = \lambda_{d,n}^2 Z_n^{d,\mathrm{Dir}}.$$

- The second example is the sequence of zonal eigenfunctions of the round sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, for $d \geq 2$, with respect to the Laplace–Beltrami operator $-\boldsymbol{\Delta}$. It is usual to consider the zonal eigenfunctions around a point, for instance $P = (1, 0, \ldots, 0)$. With such a formalism, the sequence of zonal eigenfunctions, denoted here by $Z_n^{\mathbb{S}^d}$ can be defined via the orthogonal Jacobi polynomials (or also Gegenbauer polynomials)

$$Z_n^{\mathbb{S}^d}(x) = c'_{d,n} P_n^{\left(\frac{d}{2}-1, \frac{d}{2}-1\right)}(x_1), \quad \forall x = (x_1, \ldots, x_{d+1}) \in \mathbb{S}^d, \tag{2}$$

  in which $c'_{d,n} \simeq \sqrt{n}$ (for $n \to +\infty$) and $\|Z_n^{\mathbb{S}^d}\|_{L^2(\mathbb{S}^d)} = 1$. We also have

$$-\boldsymbol{\Delta} Z_n^{\mathbb{S}^d} = n(n+d-1) Z_n^{\mathbb{S}^d}. \tag{3}$$

  Furthermore, $Z_n^{\mathbb{S}^d}$ concentrates around the point P but also around the point $-P$ thanks to the formula $P_n^{(\frac{d}{2}-1,\frac{d}{2}-1)}(-x) = (-1)^n P_n^{(\frac{d}{2}-1,\frac{d}{2}-1)}(x)$.

It is known that these models are very similar (for instance, via their L^p bounds) and more precisely that the first example is more or less a sort of canonical model (see Section 2). Let us, moreover, fix a sequence $(g_n)_{n\geq 1}$ of i.i.d. Gaussian random variables $\mathcal{N}(0,1)$. We shall prove the following result whose main contribution is the proof of the implication i) $\Rightarrow$ ii).

Theorem 1.1. — *We assume the dimension d fulfills $d \geq 2$. For simplicity, let us write $Z_n^d : \mathcal{M} \to \mathbb{R}$ one of the previous two models ($Z_n^{d,\mathrm{Dir}}$ or $Z_n^{\mathbb{S}^d}$) for which $\mathcal{M}$ is understood as the corresponding manifold of dimension d ($B_d(0,1)$ or $\mathbb{S}^d$). Let us fix a sequence $(c_n)_{n\geq 1}$ of coefficients, then the following three statements are equivalent.*

i) *The series $\sum_{n\geq 1} |c_n|^2 n^{d-1}$ is convergent.*

ii) *With probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d$ uniformly converges on $\mathcal{M}$.*

iii) *With probability 1, the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d$ weakly converges in the following sense to a function $f^{G,\omega}$, which belongs to $L^\infty(\mathcal{M})$,*

$$\forall \psi \in \mathcal{C}^\infty(\mathcal{M}) \qquad \int_{\mathcal{M}} \Big(\sum_{n=1}^{N} g_n(\omega) c_n Z_n^d(x) \Big) \psi(x) \mathrm{d}x \xrightarrow[N \to +\infty]{} \int_{\mathcal{M}} f^{G,\omega}(x) \psi(x) \mathrm{d}x.$$

Let us make a few comments on the previous result.

- Theorem 1.1 remains true by replacing the sequence (g_n) with a sequence (ε_n) of i.i.d. Rademacher random variables. Actually, if ii) holds true, then the famous contraction principle[6] shows that the Rademacher random series $\sum_{n\geq 1} \varepsilon_n(\omega) c_n Z_n^d$ almost surely uniformly converges on $\mathcal{M}$ and, thus, almost surely weakly converges in a similar sense to the assertion iii). And it turns out that the proof of iii) $\Rightarrow$ i) developed in Section 16 would be totally similar in the Rademacher case.
- Let us explain why the condition in i) is the best one that we could expect. For simplicity, let us call P a point of concentration of each Z_n^d (even if P is the origin for the model $Z_n^d = Z_n^{d,\mathrm{Dir}}$). Due to such a concentration, we may expect that the behavior of the Gaussian random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d$ is merely relevant at P, namely if the Gaussian numerical random series $\sum_{n\geq 1} g_n(\omega) c_n Z_n^d(P)$ is convergent. But, it is a basic fact that, for any complex sequence $(a_n)_{n\geq 1}$, the Gaussian random series $\sum g_n(\omega) a_n$ is almost surely convergent if and only if $\sum_{n\geq 1} |a_n|^2 < +\infty$. As a consequence, the implication ii) $\Rightarrow$ i) is obvious since we have[7] $Z_n^d(P) \simeq n^{\frac{d-1}{2}}$. The implication iii) $\Rightarrow$ i) is not difficult either (see Section 16) from a probabilistic point of view (though the proof needs a few basic facts of semi-classical analysis and heat kernel theory).

6. See, for instance, [30, page 137, Th IV.4], [28, page 99, Lem 4.5] or [32, page 45, Th 4.9]. We also refer to a result of Hoffman–Jorgensen as stated in [22, Theorem 5.2].

7. See (21) below.