Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 152
Fascicule 3

2024

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 547-569

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 152, septembre 2024

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 160 € ($ 240),
avec supplément papier : Europe 244 €, hors Europe 330 € ($ 421)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
152 (3), 2024, p. 547-569

θ-TYPE FRACTIONAL MARCINKIEWICZ INTEGRALS AND THEIR COMMUTATORS ON NON-HOMOGENEOUS GRAND SPACES

BY GUANGHUI LU, MIAOMIAO WANG & SHUANGPING TAO

ABSTRACT. — Let (X, d, μ) be a non-homogeneous space satisfying growth conditions. Under the assumption that the measure $\mu(X)$ is finite, the authors prove that a θ-fractional Marcinkiewicz integral $\mathcal{M}_{\rho,\beta,m,\theta}$ is bounded on the grand Lebesgue space $L^{p),\tau}_{\mu}(X)$ and on the grand Morrey space $M^{p),r,\tau}_{\mu}(X)$. Furthermore, the boundedness of the commutator $\mathcal{M}_{\rho,\beta,m,\theta,b}$ generated by $b \in \mathrm{Lip}_{\beta}(\mu)$ (or $b \in \mathrm{RBMO}(\mu)$) and $\mathcal{M}_{\rho,\beta,m,\theta}$ on spaces $L^{p),\tau}_{\mu}(X)$ and on spaces $M^{p),r,\tau}_{\mu}(X)$ is proved.

RÉSUMÉ (*Opérateurs intégraux fractionnaires de θ-type Marcinkiewicz et leurs échangeurs sur espaces non-homogène grand*). — Soit (X, d, μ) un espace hétérogènes répondant aux conditions de croissance. Dans l'hypothèse d'une mesure $\mu(X)$ limitée, les auteurs ont montré que l'opérateur intégré $\mathcal{M}_{\rho,\beta,m,\theta}$ de Marcinkiewicz de type θ est finitude sur un espace Lebesgue maximal principal $L^{p),\tau}_{\mu}(X)$ et un espace Morrey maximal principal $M^{p),r,\tau}_{\mu}(X)$. Va plus loin et la limitation dans l'espace de $L^{p),\tau}_{\mu}(X)$ et $M^{p),r,\tau}_{\mu}(X)$ pour les échangeurs $\mathcal{M}_{\rho,\beta,m,\theta,b}$ générés par $b \in \mathrm{Lip}_{\beta}(\mu)$ (ou $b \in \mathrm{RBMO}(\mu)$) et $\mathcal{M}_{\rho,\beta,m,\theta}$ est démontrée.

Texte reçu le 24 juillet 2023, modifié le 29 février 2024, accepté le 21 mars 2024.

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Mathematical subject classification (2010). — 42B20, 42B35, 46E30, 47B47.

Key words and phrases. — Non-homogeneous space, θ-type fractional Marcinkiewicz integral, commutator, space $\mathrm{RBMO}(\mu)$, grand Morrey space.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2024/547/$ 5.00

doi:10.24033/bsmf.2895

1. Introduction

In 1992, Iwaniec and Sbordone [11] introduced a new class of function spaces $L^{p)}(\Omega)$, when they studied the integrability of the Jacobian in a bounded open set $\Omega \subset \mathbb{R}^n$. Compared with the classical Lebesgue spaces on Euclidean spaces, these spaces are much larger, and they are called grand Lebesgue spaces. In 1997, Greco et al. [6] obtained a generalized version of grand Lebesgue spaces and denoted them by $L^{p),\theta}(\Omega)$, where $1 < p < \infty$ and $\theta > 0$. Since then, many papers have focused on the bounded properties of various integral operators on grand Lebesgue spaces and other grand spaces. For example, Fiorenza et al. [3] gave the definition of a weighted grand Lebesgue space $L^{p)}_{\omega}(0,1)$ for $p \in (1,\infty)$ and got the necessary and sufficient conditions for Hardy–Littlewood maximal operators on spaces $L^{p)}_{\omega}(0,1)$. In 2011, Meskhi [28] introduced a grand Morrey space $L^{p),\theta,\lambda}(X,\mu)$, where $p \in (1,\infty)$, $\theta > 0$ and $\lambda \in [0,1)$, and then proved that Hardy–Littlewood maximal operators, potentials and singular integrals are bounded on spaces $L^{p),\theta,\lambda}(X,\mu)$. In 2015, Meskhi [29] showed that the fractional integral operators I_α and fractional Maximal functions M_α are bounded on weighted grand Lebesgue spaces $L^{p),\theta}_{\omega}(\Omega)$. Recently, Lu [22] gave the definition of a grand generalized weighted Morrey space $L^{p),\varphi,\Phi}_{\omega}(X)$ over RD-spaces and also established the boundedness of Hardy–Littlewood maximal operators M on spaces $L^{p),\varphi,\Phi}_{\omega}(X)$. For more research on the various grand spaces and their applications, we refer readers to [1, 2, 4, 8, 15, 14, 26, 30] and the references therein.

In 2014, Sawano and Yabuta [32] gave the definition of fractional type Marcinkiewicz integral operators $\mathcal{M}_{\Omega,\rho,\alpha,q}$ associated to surfaces and also proved that $\mathcal{M}_{\Omega,\rho,\alpha,q}$ is bounded from the Triebel–Lizorkin spaces $\dot{F}^{\alpha}_{pq}(\mathbb{R}^n)$ into Lebesgue spaces $L^p(\mathbb{R}^n)$. Since these operators extend the properties of the classical Marcinkiewicz integral operators and parameter Marcinkiewicz integral operators, the bounded properties of fractional type Marcinkiewicz integral operators on various function spaces are widely focused on; for example, see [7, 16, 21, 24, 33, 35]. Moreover, Yabuta [36] first introduced an θ-type Calderón–Zygmund operator and obtained its boundedness on Lebesgue spaces. Since then, many authors focus the properties of the θ-type integral operators on various function spaces. For example, in 2020, Lu [20] proved that the parameter θ-type Marcinkiewicz integral $\mathcal{M}^{\rho}_{\theta}$ is bounded from generalized weighted Morrey spaces $L^{p,\Phi,\tau}(\omega)$ into generalized weighted weak Morrey spaces $WL^{p,\Phi,\tau}(\omega)$. In 2021, Lu and Rui [23] showed that the θ-type generalized fractional integral $T_{\alpha,\theta}$ and its commutator $T_{\alpha,\theta,b}$ formed by $b \in \mathrm{RBMO}(\mu)$ and the $T_{\alpha,\theta}$ are bounded on non-homogeneous variable exponent spaces $M^{p(\cdot)}_{q(\cdot)}(X)$. In 2023, Lu et al. [25] proved that the θ-type Calderón–Zygmund operator $\widetilde{T}_{\theta}$ and its commutator $\widetilde{T}_{\theta,b}$ associated with BMO functions are bounded on (grand) generalized weighted variable exponent Morrey space over RD-spaces.

But, in this paper, we mainly consider the boundedness of θ-type fractional type Marcinkiewicz integrals $\mathcal{M}_{\rho,\beta,m,\theta}$ and their commutators $\mathcal{M}_{\rho,\beta,m,\theta,b}$ on grand Lebesgue spaces and grand Morrey spaces over non-homogeneous spaces satisfying a growth condition.

Before stating the organization of this paper, we need to recall some necessary notions. The following definition of a quasi-metric measure space is from [13].

DEFINITION 1.1. — Let (X, d, μ) be a quasi-metric measure space (non-homogeneous space); that is, X is an abstract set, $d : X \times X \to [0,\infty)$ is a function satisfying the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there exists some constant $\kappa \geq 1$ such that, for all $x, y, z \in X$, the following inequality

$$d(x, y) \leq \kappa[d(x, z) + d(y, z)]$$

holds; μ is a measure defined on X such that all balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are measurable with meeting the growth condition, i.e., there exists a positive constant C such that, for all $x \in X$ and ball $B(x, r)$, the following inequality

$$\mu(B(x, r)) \leq Cr \tag{1}$$

holds. Moreover, in this paper, we always assume that $\mu(\{x\}) = 0$ for all $x \in X$.

For any ball B, we respectively denote its center and radius by c_B and r_B (or $r(B)$). Let $\alpha, \beta \in (1,\infty)$; a ball B is said to be an (α, β)-doubling ball if $\mu(\alpha B) \leq \beta\mu(B)$, where αB denotes the ball with the same center as B and $r(\alpha B) = \alpha r(B)$. Moreover, for any given ball B, we denote by $\widetilde{B}$ the smallest doubling ball that contains B and has the same center with B. For any two balls $B \subset S \subset X$, define

$$K_{B,S} = 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(2^k B)}{r(2^k B)}, \tag{2}$$

where $N_{B,S}$ denotes the smallest integer k such that $r(2^k B) \geq r(S)$.

We now recall the following definition of the RBMO space introduced in [34].

DEFINITION 1.2. — Let $\tau > 1$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\mathrm{RBMO}(\mu)$ if there exists a constant $C > 0$ such that, for any ball B

centered at some point of $\mathrm{supp}(\mu)$,

$$\frac{1}{\mu(\tau B)}\int_B |f(x)-m_{\widetilde{B}}(f)|\mathrm{d}\mu(x)\le C, \tag{3}$$

and, for any two balls B and S such that $B\subset S$,

$$|m_B(f)-m_S(f)|\le CK_{B,S}, \tag{4}$$

where $m_B(f)$ represents the mean value of the function f over the ball B, i.e.,

$$m_B(f)=\frac{1}{\mu(B)}\int_B f(y)\mathrm{d}\mu(y).$$

Moreover, the minimal constant C satisfying (3) and (4) is defined to be the norm of f in the space $\mathrm{RBMO}(\mu)$ and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

The notion of a θ-type fractional type Marcinkiewicz integral is as follows.

Definition 1.3. — Let θ be a non-negative and non-decreasing function on $(0,\infty)$ and satisfy

$$\int_0^1 \frac{\theta(t)}{t}\log\left(\frac{1}{t}\right)\mathrm{d}t<\infty. \tag{5}$$

A function $K_\theta(\cdot,\cdot)\in L^1_{\mathrm{loc}}(X^2\setminus\{(x,x):x\in X\})$ is called an θ-type kernel if there exists some positive constant C such that

$$|K_\theta(x,y)|\le\frac{C}{[d(x,y)]^{1-\beta}}, \tag{6}$$

and, for all $x,x',y\in X$ with $d(x,y)\ge 2d(x,x')$,

$$|K_\theta(x,y)-K_\theta(x',y)|+|K_\theta(y,x)-K_\theta(y,x')|\le C\theta\left(\frac{d(x,x')}{d(x,y)}\right)\frac{[d(x,x')]^\beta}{d(x,y)}, \tag{7}$$

where $\beta\ge 0$.

Remark 1.4. — If we take $(X,d,\mu)=(\mathbb{R}^n,d,\mu)$, $\beta=0$ and $\theta(t)=t^\delta$ with $\delta\in(0,1)$ in Definition 1.4, then the θ-type kernel K_θ is just the Calderón–Zygmund kernel K introduced in [10, 37].

Let $L^\infty_b(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A sublinear operator $\mathcal{M}_{\rho,\beta,m,\theta}$ is called an **θ-type fractional Marcinkiewicz integral operator** with the kernel K_θ satisfying (6) and (7) if, for all $f\in L^\infty_b(\mu)$ and $x\in\big(X\setminus\mathrm{supp}(f)\big)$,

$$\mathcal{M}_{\rho,\beta,m,\theta}(f)(x)=\left(\int_0^\infty\left|\frac{1}{t^{\beta+\rho}}\int_{d(x,y)<t}\frac{K_\theta(x,y)}{[d(x,y)]^{-\rho}}f(y)\mathrm{d}\mu(y)\right|^m\frac{\mathrm{d}t}{t}\right)^{\frac{1}{m}}, \tag{8}$$

where $\rho\ge 0$, $\beta\ge 0$ and $m>1$.

REMARK 1.5. — (i) If we take $(X, d, \mu) = (\mathbb{R}^n, d, \mu)$, $\beta = 1$, $\rho = 0$, $m = 2$ and $\theta(t) = t^\delta$ with $\delta \in (0, 1)$, then the θ-type fractional Marcinkiewicz integral $\mathcal{M}_{\rho,\beta,m,\theta}$ defined as in (8) is just the Marcinkiewicz integral $\mathcal{M}$ introduced by Hu et al. in [9].

(ii) If we take $(X, d, \mu) = (\mathbb{R}^n, d, \mu)$, $\beta = 0$, $m = 2$ and $\theta(t) = t^\delta$ with $\delta \in (0, 1)$, then the θ-type fractional Marcinkiewicz integral $\mathcal{M}_{\rho,\beta,m,\theta}$ defined as in (8) is just the parameter Marcinkiewicz integral $\mathcal{M}_\rho$ being sightly modified from [17, 27].

(iii) If we take $\beta = 0$, $m = 2$ and $\rho = 0$ in (8), then the θ-type fractional Marcinkiewicz integral $\mathcal{M}_{\rho,\beta,m,\theta}$ is just the θ-type Marcinkiewicz integral $\mathcal{M}_\theta$ introduced in [18].

Given $b \in \mathrm{RBMO}(\mu)$, the commutator $\mathcal{M}_{\beta,\rho,m,\theta,b}$ formed by b and the $\mathcal{M}_{\beta,\rho,m,\theta}$ is defined by

$$\begin{aligned}(9)\qquad &\mathcal{M}_{\rho,\beta,m,\theta,b}(f)(x)\\ &= \left(\int_0^\infty \left| \frac{1}{t^{\beta+\rho}} \int_{d(x,y)<t} \frac{K_\theta(x,y)}{[d(x,y)]^{-\rho}} (b(x) - b(y)) f(y) \mathrm{d}\mu(y) \right|^m \frac{\mathrm{d}t}{t} \right)^{\frac{1}{m}}.\end{aligned}$$

Let $p \in (1, \infty)$, and $\varphi(\cdot)$ be a positive bounded function defined on $(0, p-1]$ and satisfying $\lim_{x\to 0} \varphi(x) = 0$. For convenience, the class of such functions is denoted by Φ_p. On the basis of this, we now recall the following definitions of a grand Lebesgue space and a grand Morrey space introduced in [13].

DEFINITION 1.6. — Let $p \in (1, \infty)$, $\varphi \in \Phi_p$ and $\mu(X) < \infty$. Then the grand Lebesgue space $L^{p),\varphi}_\mu(X)$ is defined by

$$L^{p),\varphi}_\mu(X) = \left\{ f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{L^{p),\varphi}_\mu(X)} < \infty \right\},$$

where

$$(10)\qquad \|f\|_{L^{p),\varphi}_\mu(X)} = \sup_{0<\varepsilon<p-1} [\varphi(\varepsilon)]^{\frac{1}{p-\varepsilon}} \left(\int_X |f(x)|^{p-\varepsilon} \mathrm{d}\mu(x) \right)^{\frac{1}{p-\varepsilon}}.$$

In particular, if we take $\varphi(\varepsilon) = \varepsilon^\tau$ with $\tau \in (0, \infty)$ in (10), then the grand Lebesgue space $L^{p),\varphi}_\mu(X)$ is simply denoted by

$$L^{p),\varphi}_\mu(X) = L^{p),\tau}_\mu(X).$$

DEFINITION 1.7. — Let $1 < p \leq r < \infty$, $\ell > 0$, $\varphi \in \Phi_p$ and $\mu(X) < \infty$. Then the grand Morrey space $M^{p),r,\varphi}_{\mu,\ell}(X)$ is defined by

$$M^{p),r,\varphi}_{\mu,\ell}(X) = \left\{ f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{M^{p),r,\varphi(\cdot)}_{\mu,\ell}(X)} < \infty \right\},$$