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BRANCHING CAPACITY OF A RANDOM WALK RANGE

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ABSTRACT. — We consider the branching capacity of the range of a simple random walk on $\mathbb{Z}^d$, with $d \geq 5$ and show that it falls within the same universality class as the volume and the capacity of the range of simple random walks and branching random walks. To be more precise we prove a law of large numbers in dimension $d \geq 6$, with a logarithmic correction in dimension 6 and identify the correct order of growth in dimension 5. The main original part is the law of large numbers in dimension 6, for which one needs a precise asymptotic of the non-intersection probability of an infinite invariant critical tree-indexed walk with a two-sided simple random walk. The result is analogous to the estimate proved by Lawler for the non-intersection probability of an infinite random walk with a two-sided walk in dimension 4. While the general strategy of Lawler's proof still applies in this new setting, many steps require new ingredients.

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Résumé (*Capacité branchante de la trace d'une marche aléatoire*). — Nous considérons la capacité branchante de la trace d'une marche aléatoire simple dans $\mathbb{Z}^d$, avec $d \geq 5$. Nous montrons que cette fonctionnelle est dans la même classe d'universalité que le volume et la capacité de la trace d'une marche aléatoire simple ou d'une marche branchante. Plus précisément nous montrons une loi des grands nombres en dimension $d \geq 6$, avec une correction logarithmique en dimension 6, and nous identifions l'ordre de grandeur de cette fonctionnelle en dimension 5. La partie la plus originale de notre travail concerne la dimension 6, pour laquelle nous avons besoin de montrer une asymptotique précise de la probabilité de non-intersection d'une marche aléatoire indexée par un arbre critique invariant infini avec une marche aléatoire simple bi-directionnelle. C'est l'analogue d'un résultat célèbre de Lawler sur la probabilité de non-intersection d'une marche aléatoire simple avec une marche bi-directionnelle en dimension 4. L'idée générale de la preuve est la même dans les deux cas, mais de nombreuses étapes nécessitent de nouveaux ingrédients dans ce nouveau cadre.

1. Introduction

We start by recalling some important definitions and we then state our main results. The branching capacity is defined here in terms of an offspring distribution μ on $\mathbb{N}$, which is fixed in the whole paper and assumed to be critical, in the sense that $\sum_i i\mu(i) = 1$. We further assume that it has a finite and positive variance σ^2. We write the size biased distribution of μ as μ_{sb}, which we recall is defined by $\mu_{\text{sb}}(i) = i\mu(i)$, for all $i \geq 0$.

We then consider $\mathcal{T}$ an infinite planar tree, introduced independently in [19] and [4], which generalizes the one-sided version of Le Gall and Lin [16], and which is defined as follows (here the offspring of every vertex are ordered from left to right, and the root is at the bottom of the tree):

- The root produces i offspring with probability $\mu(i-1)$ for every $i \geq 1$. The first offspring of the root is *special*, while the others, if they exist, are *normal*.
- Special vertices produce offspring independently according to μ_{sb}, while normal vertices produce offspring independently according to μ.
- One of the offspring of a special vertex is chosen at random to be a special vertex, while the rest are normal ones.

Since μ-trees are almost surely finite, because μ is critical, and since special vertices are guaranteed to have at least one offspring by definition of μ_{sb}, $\mathcal{T}$ has a unique infinite path emanating from the root that we call *spine*; it contains the root together with all the special vertices. We assign label 0 to the root. We assign positive labels to the vertices to the right of the spine according to depth first search from the root and we assign negative labels to the vertices to the left of the spine and the spine vertices, as well according to depth first search from infinity; see Figure 1.1. We call the vertices with negative labels (including the spine vertices) the *past* of $\mathcal{T}$ and denote them $\mathcal{T}_-$, while the

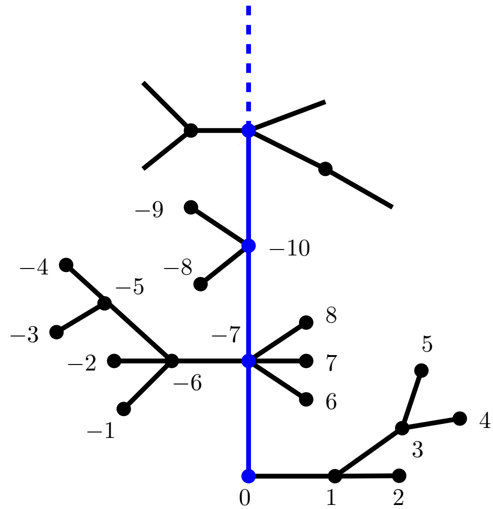

Figure 1.1. An infinite tree $\mathcal{T}$, with the spine in blue

vertices with non-negative labels are in the *future* of $\mathcal{T}$, and we denote them $\mathcal{T}_+$. Note that the root does not have any offspring in the past of $\mathcal{T}$.

Given $x \in \mathbb{Z}^d$, we denote by $(S_u^x)_{u\in\mathcal{T}}$ the random walk indexed by $\mathcal{T}$, starting from x, whose jump distribution is the uniform measure on the neighbors of the origin, and denote its range in the past by

$$\mathcal{T}_-^x = \{S_u^x : u \in \mathcal{T}_-\}.$$

The equilibrium measure e_A of a finite set $A \subset \mathbb{Z}^d$, with $d \geq 5$, has been introduced by Zhu [18], and is defined by,

$$e_A(x) = \mathbb{P}(\mathcal{T}_-^x \cap A = \emptyset).$$

Then the branching capacity of a finite set A is defined similarly to the usual Newtonian capacity, namely

$$\mathrm{BCap}(A) = \sum_{x\in A} e_A(x).$$

Consider now $(X_n)_{n\geq 0}$ an independent simple random walk on $\mathbb{Z}^d$ (i.e., a random walk whose law of increments is the uniform measure on the neighbors of the origin), and define its range at time n as

$$\mathcal{R}_n = \{X_0, \dots, X_n\}.$$

Our main object of study in this paper is the branching capacity of the range $\mathrm{BCap}(\mathcal{R}_n)$, in dimension $d \geq 5$, and our goal is to show that it satisfies the same universal asymptotic behavior as the volume [9] and the capacity [1, 2, 8, 10] of the range, with only a shift of the critical dimension of, respectively, two and four units, which is here the dimension 6. Interestingly, the same universal results have also been proved recently for the volume [15, 16] and the capacity [4, 5] of a critical branching random walk, and of course it would be of interest to see if they can also be extended to the branching capacity of a branching random walk, but we leave this for a future work.

Our first result is a strong law of large numbers. The proof is entirely similar to that for the usual Newtonian capacity, which dates back to Jain and Orey [10], and is reproduced at the end of this paper for reader's convenience (to be more precise, the fact that the limiting constant is positive requires a specific argument).

THEOREM 1.1. — *Assume $d \geq 7$. There exists a constant $c_d > 0$, such that almost surely,*

$$\lim_{n\to\infty} \frac{\mathrm{BCap}(\mathcal{R}_n)}{n} = c_d. \tag{1}$$

It is very likely that a central limit theorem, with the usual renormalization in $\sqrt{n}$, could be proved in dimension $d \geq 8$, following the same lines as in [1]. In dimension 7 it is expected that a logarithmic correction should appear in the normalization, but this might be a much more challenging problem, as the corresponding results in the simpler cases of the volume and the capacity of the range of a random walk are already quite involved; see [11, 17], respectively.

The main contribution of this paper is the law of large numbers in dimension 6, which requires some more original work. We only present here a detailed proof of the weak law (with a convergence in probability), but a strong law (with an almost sure convergence) could be proved as well without much additional work; see Remark 3.6 for more details. The main step is to obtain the asymptotic of the expected branching capacity of the range. The general strategy for this is the same as for the capacity of the range, in which case the corresponding result follows from the estimates proved by Lawler [12] for the non-intersection probability between one walk and another independent two-sided walk in dimension 4; see [1, 8]. However, one serious issue that arises when working with the tree-indexed walk is the lack of Markov property, which in particular has the damaging consequence that there is no simple last exit formula as one has for a simple random walk. This leads to some non-trivial complications, which can fortunately be overtaken.

THEOREM 1.2. — *Assume $d = 6$, and that μ has a finite third moment. Then one has the convergence in probability and in L^2,*

$$\lim_{n\to\infty} \frac{\log n}{n} \cdot \mathrm{BCap}(\mathcal{R}_n) = \frac{2\pi^3}{27\sigma^2}.$$

Of course, a natural question now would be to prove a central limit theorem, as was done in [14, 2], respectively, for the volume and the capacity of the range. We leave this for future work, as it would require some really new ingredients; in particular one major issue would be to identify a simple expression for the term

$$\chi(A, B) = \mathrm{BCap}(A \cup B) - \mathrm{BCap}(A) - \mathrm{BCap}(B),$$

where A and B are arbitrary finite subsets of $\mathbb{Z}^d$ and improve the bounds that we have on the variance of $\mathrm{BCap}(\mathcal{R}_n)$.

To conclude we provide bounds identifying the correct order of growth of the expected branching capacity of the range in dimension 5. The upper bound is easily obtained by using monotonicity of the branching capacity and known bounds on the branching capacity of balls. The lower bound is more difficult, and we rely here on a recent result of [3] showing a variational characterization of the branching capacity. We also mention that an invariance principle is in progress [6, 7].

Proposition 1.3. — *For $d = 5$, there exist positive constants c_5^- and c_5^+, such that for all $n \geq 1$,*

$$c_5^- \cdot \sqrt{n} \leq \mathbb{E}\left[\mathrm{BCap}(\mathcal{R}_n)\right] \leq c_5^+ \cdot \sqrt{n}.$$

Remark 1.4. — *We note that the proofs of all our results would extend immediately to any symmetric finite range jump distribution both for the tree-indexed walk and the walk $(X_n)_{n\geq 0}$. It is even likely that one would only need a moment assumption, e.g., as in* [6]. *Concerning the random walk $(X_n)_{n\geq 0}$, the hypothesis of symmetric jump distribution could be relaxed to a centered jump distribution.*

The paper is organized as follows. In Section 2, we prove some preliminary results that could be of general interest. In particular, we prove an analogous version in the setting of branching random walks of a key equation discovered by Lawler, relating some non-intersection events and a sum of Green's function along the positions of a random walk; see Lemma 2.7 and Corollary 2.8. We also prove there some quantitative bounds on the speed of convergence toward the branching capacity of a set A, of the (conveniently normalized) probability to hit A for a tree-indexed walk, as the starting point goes to infinity; see Proposition 2.6. Then Section 3 focuses on the case of dimension 6, and we prove there Theorem 1.2, while short proofs of Theorem 1.1 and Proposition 1.3 are given in Section 4.